On ground fields of arithmetic hyperbolic reflection groups

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Abstract

Using authors's methods of 1980, 1981, some explicit finite sets of number fields containing ground fields of arithmetic hyperbolic reflection groups are defined, and good bounds of their degrees (over $\mathbb{Q}$) are obtained. For example, degree of the ground field of any arithmetic hyperbolic reflection group in dimension at least 6 is bounded by 120. These results could be important for further classification.

We also formulate a mirror symmetric conjecture to finiteness of the number of arithmetic hyperbolic reflection groups which was established in full generality recently.

This paper also gives corrections to my papers [17] (see appendix) and [34].

Dedicated to John McKay

1 Introduction

There are only three types of simply-connected complete Riemannian manifolds of constant curvature: spheres, Euclidean spaces and hyperbolic spaces. Discrete reflection groups (generated by reflections in hyperplanes in these spaces) were defined by H.S.M. Coxeter. He classified these groups in spheres and Euclidean spaces [8].

There are two types of discrete reflection groups with fundamental domain of finite volume in hyperbolic spaces: general and arithmetic. In this paper, we consider only arithmetic hyperbolic reflection groups.

In [27], Vinberg (1967) stated and proved a criterion for the arithmeticity of discrete reflection groups in hyperbolic spaces in terms of their fundamental chambers. In particular, he introduced the notion of the ground field of such groups. This is a totally real algebraic number field $\mathbb{K}$ of a finite degree over

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$\mathbb{Q}$. The arithmetic reflection group W is a subgroup of finite index of the automorphism group $O(S)$ of a hyperbolic quadratic form (a hyperbolic lattice) S over the ring of integers $\mathbb{O}$ of this field. See Sect. 6 for the exact definition. Hyperbolic lattices S having a reflection subgroup $W \subset O(S)$ of finite index are called *reflective*. One can canonically associate to S the hyperbolic space $\mathcal{L}(S)$ of dimension $\operatorname{rank} S - 1$ such that groups $W \subset O^+(S)$ act in $\mathcal{L}(S)$ and define the arithmetic hyperbolic reflection group W. One can always take the maximal reflection subgroup $W(S) \subset O(S)$, the reflection group of the lattice S, which contains W. Thus, S is reflective if and only if $[O(S) : W(S)] < \infty$. Two hyperbolic lattices which differ by multiplication of their forms by $k \in \mathbb{K}$ are called *similar*. Their hyperbolic spaces and automorphism groups are clearly identified.

Thus, *classification of similarity classes of reflective hyperbolic lattices* is the key problem in classification of arithmetic hyperbolic reflection groups. It includes classification of maximal arithmetic hyperbolic reflection groups, and it is important for classification of hyperbolic Lie algebras. The degree $N = [\mathbb{K} : \mathbb{Q}]$ of the ground field $\mathbb{K}$ of S and W, and the dimension $n = \dim \mathcal{L}(S) = \operatorname{rank} S - 1 \geq 2$ are the most important parameters for this classification.

In [16, 17], the author proved (1980, 1981) that the number of similarity classes of reflective hyperbolic lattices is finite for the fixed parameters n and N. Moreover, the number of ground fields of the fixed degree N is also finite. In [17], the author proved (1981) that there exists a effective constant N_0 such that $N \leq N_0$ if the dimension $n \geq 10$.

In [28], [29], Vinberg proved (1981) that $n < 30$: arithmetic hyperbolic reflection groups don't exist in dimensions $n \geq 30$.

Thus, the numbers of similarity classes of reflective hyperbolic lattices and maximal arithmetic hyperbolic reflection groups are finite in dimensions $n \geq 10$.

About these results, see also reports [30] and [18] at International Congresses of Mathematicians.

Almost 25 years boundedness of degree $N = [\mathbb{K} : \mathbb{Q}]$ remained opened in small dimensions $2 \leq n \leq 9$. Only in 2005 it was proved in dimension $n = 2$ by Long, Maclachlan and Reid [15] and in dimension $n = 3$ by Agol [1]. In 2006, the author shown [20] that the boundedness in remaining dimensions $4 \leq n \leq 9$ can be easily deduced from boundedness in dimensions $n = 2$, 3 and methods of [16] and [17] (see Theorem 12 and its proof in Sect. 4.3).

Thus, now, finiteness of the numbers of reflective hyperbolic lattices and of maximal arithmetic hyperbolic reflection groups are established in full generality: for all dimensions of hyperbolic spaces and for all ground fields together.

Unfortunately, these finiteness results are very far from classification of these finite sets. The purpose of this paper, is to prove some explicit results in this direction. Perhaps, the first and the most important problem is to enumerate possible ground fields $\mathbb{K}$ and their degrees $[\mathbb{K} : \mathbb{Q}]$.

First explicit results in this direction were obtained by Vinberg [28], [29] in 1981. He had shown: for dimensions $n \geq 30$, the set of ground fields is empty; for dimensions $n \geq 22$, ground fields belong to union of $\mathcal{F}L^4 = \{\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})\}$ and $\{\mathbb{Q}(\cos(2\pi/7))\}$; for dimensions $n \geq 14$ ground fields belong to the set $\mathcal{F}T$ of

13 fields which are ground fields of arithmetic triangle (plane) reflection groups classified by Takeuchi [23], [24] in 1977 (their degree is bounded by 5). See Sect. 3.2 about arithmetic triangle groups.

Our results can be considered as some extensions of these explicit Vinberg's results to smaller dimensions.

In Sect. 3.3, we introduce finite sets of fields $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1,2,3,4,5$. They are ground fields of some V-arithmetic 3-dimensional fundamental edge chambers of minimality 14 described by their connected hyperbolic Gram graphs $\Gamma_i^{(4)}$, $i = 1,2,3,4,5$, with four vertices in Figures 3 — 8. They give some special types of V-arithmetic edge chambers introduced and used in [17]. Using methods of [17], in Theorem 4 and Sect. 5 we show that the degrees of fields from these sets are bounded by reasonable constants 24, 39, 53, 120 and 120 respectively.

Following methods of [16], [17] and [29], in Theorem 5 we show that in dimensions $n \geq 10$, the ground field of any arithmetic hyperbolic reflection group belongs to one of sets: $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1,2,3,4$. In particular, the degree is bounded by 120. Thus, author's result in [17] becomes very explicit: the constant N_0 which we mentioned above can be taken to be $N_0 = 120$.

In Sect. 4.2, we introduce one more set $\mathcal{F}_{2,4}(14)$ of fields. It is the set of ground fields of arithmetic quadrangles of minimality 14 (i. e. ground fields of 2-dimensional arithmetic hyperbolic reflection groups with quadrangle fundamental polygon of minimality 14). According to Takeuchi [25], the set of ground fields of arithmetic quadrangles is finite, and degrees of these fields are bounded by 11.

Following methods of [16], [17], in Theorem 9 we show that in dimensions $n \geq 6$ the ground field of any arithmetic hyperbolic reflection group belongs to one of sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1,2,3,4,5$ and $\mathcal{F}_{2,4}(14)$. In particular, its degree is bounded by 120.

Unfortunately, now we don't have so explicit results for smaller dimension. Following [20], we only show in Theorem 12 that in dimensions $n \geq 4$, the ground field of any arithmetic hyperbolic reflection group belongs to one of sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1,2,3,4$, or it is the ground field of 3 or 2-dimensional arithmetic hyperbolic reflection group with a fundamental chamber of minimality 14. Thus, the degree is bounded by the maximum of 120 and of degrees of ground fields of 3 and 2-dimensional arithmetic hyperbolic reflection groups of minimality 14.

For 3-dimensional arithmetic hyperbolic reflection groups, we know finiteness of the number of ground fields by Agol [1]. By author's knowledge, no explicit bound of their degree is known.

Following Long, Maclachlan and Reid [15] and Borel [3], Takeuchi [25], in Sect. 4.5 we show that degree of the ground field of any 2-dimensional arithmetic hyperbolic reflection group is bounded by 44.

Using known at that time finiteness result for hyperbolic reflective lattices over $\mathbb{Z}$, in [10]–[19] some finiteness results for IV type (i. e. of signature $(2,t)$) integer reflective lattices S were obtained, and some general conjectures were formulated. Here a lattice S over $\mathbb{Z}$ of signature $(2,t)$ is called reflective if IV

type Hermitian symmetric domain associated to S has an $O^+(S)$-automorphic holomorphic form Φ of positive weight such that all components of its divisor are quadratic divisors orthogonal to roots (giving reflections) of S. This automorphic form is called *reflective.* One can consider these finiteness statements about IV type reflective lattices over $\mathbb{Z}$ as "mirror symmetric" to finiteness results about hyperbolic reflective lattices over $\mathbb{Z}$. Since we now know finiteness of hyperbolic reflective lattices in general, in Sect. 6 we formulate the corresponding mirror symmetric conjecture about IV type reflective lattices in general — over arbitrary totally real algebraic number fields. We expect that it is valid.

Some arithmetic hyperbolic reflection groups and some reflective automorphic forms and corresponding hyperbolic and IV type reflective lattices over $\mathbb{Z}$ are important in Borcherds proof [3] of Moonshine Conjecture by Conway and Norton [7] which had been first discovered by John McKay.

We hope that similar objects over arbitrary number fields will find similar astonishing applications in the future. At least, the results and conjectures of this paper show that they are very exceptional even in this very general setting.

At first, the paper appeared as preprint [21] which was published in [34]. In Appendix we review and correct Section 1 of our old paper [17] which was used in these papers. The present variant takes these corrections under considerations.

2 Reminding of some basic facts about hyperbolic fundamental polyhedra

Here we remind some basic definitions and results about fundamental chambers (always for discrete reflection groups) in hyperbolic spaces and their Gram matrices. See [27], [31] and [16], [17].

We work with Klein model of a hyperbolic space $\mathcal{L}$ associated to a hyperbolic form Φ over the field of real numbers $\mathbb{R}$ with signature $(1,n)$, where $n = \dim \mathcal{L}$. Let $V = \{x \in \Phi | x^2 > 0\}$ be the cone determined by Φ, and let V^+ be one of the two halves of this cone. Then $\mathcal{L} = \mathcal{L}(\Phi) = V^+/\mathbb{R}^+$ is the set of rays in V^+; we let $[x]$ denote the element of $\mathcal{L}$ determined by the ray $\mathbb{R}^+x$ where $x \in V^+$ and $\mathbb{R}^+$ is the set of all positive real numbers. The hyperbolic distance is given by the formula

$$\rho([x],[y]) = (x\cdot y)/\sqrt{x^2y^2},\ [x],[y] \in \mathcal{L},$$

then the curvature of $\mathcal{L}$ is equal to -1.

Every half-space $\mathcal{H}^+$ in $\mathcal{L}$ determines and is determined by the orthogonal element $e \in \Phi$ with square $e^2 = -2$:

$$\mathcal{H}^+ = \mathcal{H}^+_e = \{[x] \in \mathcal{L} | x\cdot e \geq 0\}.$$

It is bounded by the hyperplane

$$\mathcal{H}^+ = \mathcal{H}^+_e = \{[x] \in \mathcal{L} | x\cdot e = 0\}$$

orthogonal to e. If two half-spaces $\mathcal{H}^+_{e_1}$, $\mathcal{H}^+_{e_2}$ where $e_1^2 = e_2^2 = -2$ have a common non-empty open subset in $\mathcal{L}$, then $\mathcal{H}_{e_1} \cap \mathcal{H}_{e_2}$ is an angle of the value ϕ where $2\cos\phi = e_1 \cdot e_2$ if $-2 < e_1 \cdot e_2 \le 2$, and the distance between hyperplanes $\mathcal{H}_{e_1}$ and $\mathcal{H}_{e_2}$ is equal to ρ where $2\,\mathrm{ch}\,\rho = e_1 \cdot e_2$ if $e_1 \cdot e_2 > 2$.

A convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is intersection of a finite number of half-spaces $\mathcal{H}^+_e$, $e \in P(\mathcal{M})$, where $P(\mathcal{M})$ are all the vectors with square -2 which are orthogonal to the faces (of the codimension one) of $\mathcal{M}$ and are directed outward. The matrix

$$A = (a_{ij}) = (e_i \cdot e_j),\ \ e_i, e_j \in P(\mathcal{M}), \tag{1}$$

is the Gram matrix $\Gamma(\mathcal{M}) = \Gamma(P(\mathcal{M}))$ of $\mathcal{M}$. It determines $\mathcal{M}$ uniquely up to motions of $\mathcal{L}$. If $\mathcal{M}$ is sufficiently general, then $P(\mathcal{M})$ generates Φ, and the form Φ is

$$\Phi = \sum_{e_i, e_j \in P(\mathcal{M})} a_{ij} X_i Y_j \ \bmod \text{Kernel}, \tag{2}$$

and $P(\mathcal{M})$ naturally identifies with a subset of Φ and defines $\mathcal{M}$.

The polyhedron $\mathcal{M}$ is a fundamental chamber of a discrete reflection group W in $\mathcal{L}$ if and only if $a_{ij} \ge 0$ and $a_{ij} = 2\cos\frac{\pi}{m_{ij}}$ where $m_{ij} \ge 2$ is an integer if $a_{ij} < 2$ for all $i \ne j$. Symmetric real matrices A satisfying these conditions and having all their diagonal elements equal to -2 are called *fundamental* (then the set $P(\mathcal{M})$ formally corresponds to indices of the matrix A). As usual, further we identify fundamental matrices with fundamental graphs Γ. Their vertices correspond to $P(\mathcal{M})$. Two different vertices $e_i \ne e_j \in P(\mathcal{M})$ are connected by the thin edge of the integer weight $m_{ij} \ge 3$ if $0 < a_{ij} = 2\cos\frac{\pi}{m_{ij}} < 2$, by the thick edge if $a_{ij} = 2$, and by the broken edge of the weight a_{ij} if $a_{ij} > 2$. In particular, the vertices e_i and e_j are disjoint if and only if $e_i \cdot e_j = a_{ij} = 2\cos\frac{\pi}{2} = 0$. Equivalently, e_i and e_j are perpendicular (or orthogonal). See some examples of such graphs in Figures 1 — 8 below.

For a real $t > 0$, we say that a fundamental matrix $A = (a_{ij})$ (and the corresponding fundamental chamber $\mathcal{M}$) *has minimality* t if $a_{ij} < t$ for all a_{ij}. Here we follow [16], [17]. Further, the minimality $t = 14$ will be especially important.

It is known that fundamental domains of arithmetic hyperbolic groups must have finite volume. Let us assume that it is valid for a fundamental chamber $\mathcal{M}$ of a hyperbolic discrete reflection group. As Vinberg had shown [27], in order for $\mathcal{M}$ to be a fundamental chamber of an arithmetic reflection group W in $\mathcal{L}$, it is necessary and sufficient that all of the cyclic products

$$b_{i_1 \dots i_m} = a_{i_1 i_2} \cdot a_{i_2 i_3} \cdots a_{i_{m-1} i_m} \cdot a_{i_m i_1} \tag{3}$$

be algebraic integers, that the field $\widetilde{\mathbb{K}} = \mathbb{Q}(\{a_{ij}\})$ be totally real, and that, for any embedding $\widetilde{\mathbb{K}} \to \mathbb{R}$ not the identity over the *ground field* $\mathbb{K} = \mathbb{Q}(\{b_{i_1 \dots i_m}\})$ generated by all of the cyclic products (3), the form (2) be negative definite.

Fundamental real matrices $A = (a_{ij})$, $a_{ij} = e_i \cdot e_j$, $e_i, e_j \in P(\mathcal{M})$ (or the corresponding graphs), with a hyperbolic form Φ in (2) and satisfying these

Vinberg's conditions will be further called *V-arithmetic* (here we don't require that the corresponding hyperbolic polyhedron $\mathcal{M}$ has finite volume). It is well-known (and easy to see; see arguments in Sect. 5.1) that a subset $P \subset P(\mathcal{M})$ also defines a V-arithmetic matrix $(e_i \cdot e_j)$, $e_i, e_j \in P$, with the same ground field $\mathbb{K}$ if the subset P is hyperbolic, i. e. the corresponding to P form (2) is hyperbolic.

3 V-arithmetic edge polyhedra

A fundamental chamber $\mathcal{M}$ (and the corresponding Gram matrix A or a graph) is called *edge chamber (matrix, graph)* if all hyperplanes $\mathcal{H}_e$, $e \in P(\mathcal{M})$, contain one of two distinct vertices v_1 and v_2 of the 1-dimensional edge v_1v_2 of $\mathcal{M}$. Assume that both vertices v_1 and v_2 are finite (further we always consider this case). Further we call this edge chambers *finite*. Assume that $\dim \mathcal{L} = n$. Then $P(\mathcal{M})$ consists of $n+1$ elements: e_1, e_2 and $n-1$ elements $P(\mathcal{M}) - \{e_1, e_2\}$. Here $P(\mathcal{M}) - \{e_1, e_2\}$ corresponds to hyperplanes which contain the edge v_1v_2 of $\mathcal{M}$. The e_1 corresponds to the hyperplane which contains v_1 and does not contain v_2. The e_2 corresponds to the hyperplane which contains v_2 and does not contain v_1. Then the set $P(\mathcal{M})$ is hyperbolic (it has hyperbolic Gram matrix), but its subsets $P(\mathcal{M}) - \{e_1\}$ and $P(\mathcal{M}) - \{e_2\}$ are negative definite (they have negative definite Gram matrix) and define Coxeter graphs. Only the element $u = e_1 \cdot e_2$ of the Gram matrix of $\mathcal{M}$ can be greater than 2. Thus, $\mathcal{M}$ will have the minimality t if and only if $u = e_1 \cdot e_2 < t$.

From considerations above, the Gram graph $\Gamma(P(\mathcal{M}))$ of an edge chamber has only one hyperbolic connected component $P(\mathcal{M})^{hyp}$ (containing e_1 and e_2) and several negative definite connected components. Gram matrix $\Gamma(P(\mathcal{M})^{hyp})$ evidently also corresponds to an edge chamber of the dimension $\#P(\mathcal{M})^{hyp} - 1$. If $\mathcal{M}$ is V-arithmetic, the ground field $\mathbb{K}$ of $\mathcal{M}$ is the same as for the hyperbolic connected component $\Gamma(P(\mathcal{M})^{hyp})$

The following result had been proved in [17].

Theorem 1. *([17, Theorem 2.3.1]) Given $t > 0$, there exists an effective constant $N(t)$ such that every V-arithmetic edge chamber of the minimality t with ground field $\mathbb{K}$ of degree greater that $N(t)$ over $\mathbb{Q}$ has the hyperbolic connected component of its Gram graph which has less than 4 elements.*

Considerations in [17] (and also [16]) also show that the set of possible ground fields $\mathbb{K}$ of hyperbolic connected components with at least 4 vertices of V-arithmetic edge chambers of minimality t is finite. Even the set of Gram graphs $\Gamma(P(\mathcal{M})^{hyp})$ of minimality t with fixed ≥ 4 number of vertices is finite. Taking this under consideration, here we want to formulate and prove more efficient variant of this theorem. We restrict by the minimality $t = 14$ to get an exact estimate for the constant $N(14)$, but the same finiteness results are valid for any $t > 0$.

To formulate this new variant, we need to introduce some fundamental edge graphs.

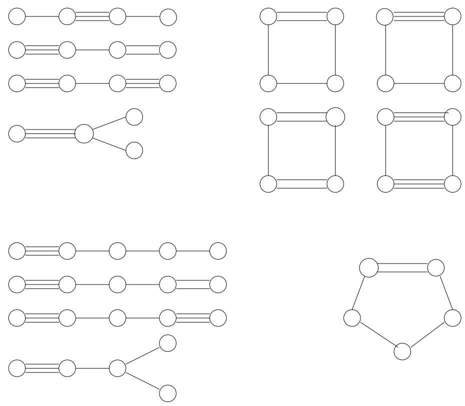

Figure 1: All arithmetic Lannér graphs with at least 4 vertices

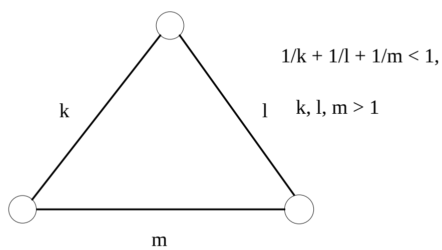

Figure 2: Triangle graphs

3.1 Arithmetic Lannér graphs with ≥ 4 elements

We remind that Lannér graphs are Gram graphs of bounded fundamental hyperbolic simplexes. They are characterized as hyperbolic fundamental graphs such that any their proper subgraph is a Coxeter graph. They were classified by Lannér [14]. In Figure 1 we give all arithmetic Lannér graphs with at least 4 vertices (only one Lannér graph with ≥ 4 vertices is not arithmetic). As usual, we replace a thin edge of the weight k by $k-2$-edges for a small k. Ground fields of Lannér graphs with ≥ 4 vertices give three fields:

$$\mathcal{F}L^4 = \{\mathbb{Q},\ \mathbb{Q}(\sqrt{2}),\ \mathbb{Q}(\sqrt{5})\}. \tag{4}$$

See [29] for details.

3.2 Arithmetic triangle graphs

Triangle graphs are Gram graphs of bounded fundamental triangles on hyperbolic plane (we don't consider non-bounded triangles). Equivalently, they are

Lannér graphs with 3 vertices. They are given in Figure 2 where $2 \leq k, l, m$ and

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1.$$

Arithmetic triangles were enumerated by Takeuchi [23]. All bounded arithmetic triangles are given by the following triplets (k, l, m):

$$(2,3,7\ -\ 12),\ (2,3,14),\ (2,3,16),\ (2,3,18),\ (2,3,24),\ (2,3,30),\ (2,4,5\ -\ 8),$$
$$(2,4,10),\ (2,4,12),\ (2,4,18),\ (2,5,5),\ (2,5,6),\ (2,5,8),\ (2,5,10),\ (2,5,20),$$
$$(2,5,30),\ (2,6,6),\ (2,6,8),\ (2,6,12),\ (2,7,7),\ (2,7,14),\ (2,8,8),\ (2,8,16),$$
$$(2,9,18),\ (2,10,10),\ (2,12,12),\ (2,12,24),\ (2,15,30),\ (2,18,18),\ (3,3,4\ -\ 9),$$
$$(3,3,12),\ (3,3,15),\ (3,4,4),\ (3,4,6),\ (3,4,12),\ (3,5,5),\ (3,6,6),\ (3,6,18),$$
$$(3,8,8),\ (3,8,24),\ (3,10,30),\ (3,12,12),\ (4,4,4\ -6),\ (4,4,9),\ (4,5,5),\ (4,6,6),$$
$$(4,8,8),\ (4,16,16),\ (5,5,5),\ (5,5,10),\ (5,5,15),\ (5,10,10),\ (6,6,6),\ (6,12,12),$$
$$(6,24,24),\ (7,7,7),\ (8,8,8),\ (9,9,9),\ (9,18,18),\ (12,12,12),\ (15,15,15).$$

Their ground fields were found by Takeuchi [24]. They give the set of fields

$$\begin{aligned}&\mathcal{F}T = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{a}) \mid a = 2,\ 3,\ 5,\ 6\} \cup \{\mathbb{Q}(\sqrt{2}, \sqrt{3}),\ \mathbb{Q}(\sqrt{2}, \sqrt{5})\} \cup \\ &\cup \{\mathbb{Q}(\cos \tfrac{2\pi}{b}) \mid b = 7,\ 9,\ 11,\ 15,\ 16,\ 20\}.\end{aligned} \tag{5}$$

3.3 V-arithmetic connected finite edge graphs with 4 vertices for $2 < u < 14$

Using classification of Coxeter graphs, it is easy to draw all possible pictures of connected finite edge graphs $\Gamma^{(4)}$ with 4 vertices and $u = e_1 \cdot e_2 > 2$. They correspond to all 3-dimensional finite fundamental edge polyhedra with connected Gram graph and $u > 2$. They are given in Figure 3 and give five types of graphs $\Gamma = \Gamma_i^{(4)}$, $i = 1, 2, 3, 4, 5$. All possible natural parameters s, k, r, $p \geq 2$ for these graphs can be easily enumerated by the condition that $\Gamma - \{e_1\}$, $\Gamma - \{e_2\}$ are Coxeter graphs. They will be given in Sec. 5 below.

Definition 2. *For $i = 1, 2, 3, 4, 5$ and $t > 0$ we denote by $\Gamma_i^{(4)}(t)$ the set of all V-arithmetic connected finite edge graphs with 4 vertices $\Gamma_i^{(4)}$ of the minimality t, i. e. for $2 < u < t$, and by*

$$\mathcal{F}\Gamma_i^{(4)}(t)$$

the set of all their ground fields.

All V-arithmetic graphs $\Gamma_i^{(4)}$ for $2 < u < t$ give particular cases of graphs of V-arithmetic edge polyhedra with hyperbolic connected component having 4 vertices and minimality t. Thus, by Theorem 1, degree (over $\mathbb{Q}$) of fields from $\mathcal{F}\Gamma_i^{(4)}(t)$ is bounded by the effective constant $N(t)$. It follows that the sets of V-arithmetic graphs $\Gamma_i^{(4)}(t)$ and fields $\mathcal{F}\Gamma_i^{(4)}(t)$ are also finite.

Vice versa, Theorem 2 can be deduced from finiteness of the sets of fields above because of the following easy statement.

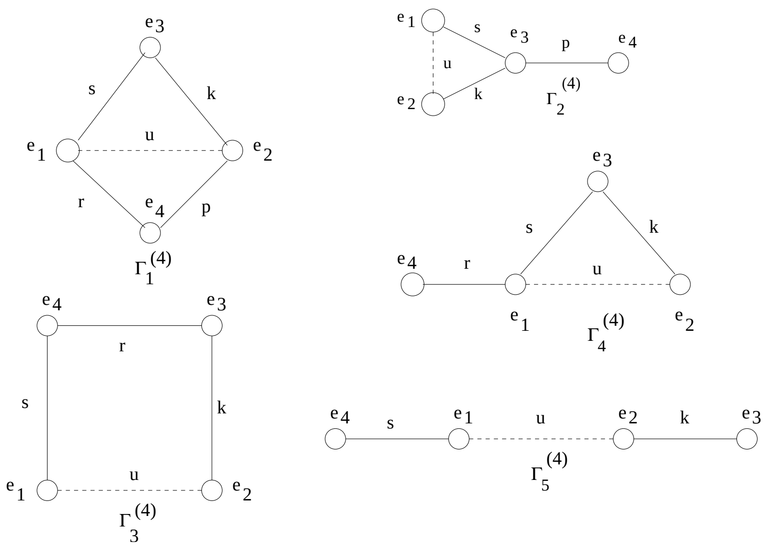

Figure 3: Five graphs $\Gamma_i^{(4)}$, $i = 1,\ 2,\ 3,\ 4,\ 5$.

Proposition 3. *The ground field of any V-arithmetic edge chamber of the minimality $t > 0$ with the hyperbolic connected component of its Gram graph having at least 4 vertices belongs to one of the finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}(t)$, $1 \le i \le 5$, introduced above.*

In particular, Theorem 1 is equivalent to finiteness of the sets of fields $\mathcal{F}\Gamma_i^{(4)}(t)$, $i = 1,\ 2,\ 3,\ 4,\ 5$.

Proof. Let $\mathcal{M}$ be a V-arithmetic edge chamber of the minimality $t > 0$, and Γ its Gram graph having the hyperbolic connected component Γ^{hyp} with at least 4 vertices. The graph Γ^{hyp} is also V-arithmetic edge graph.

Assume that $u = e_1 \cdot e_2 < 2$. Then $\{e_1, e_2\}$ gives a negative definite subgraph of Γ^{hyp}. It follows that any subgraph of Γ^{hyp} is also negative definite if it has one or two vertices. Since Γ^{hyp} is hyperbolic, it follows that Γ^{hyp} contains a minimal hyperbolic subgraph L which is Lannér with at least 3 vertices. Since L is hyperbolic, the ground field of Γ is equal to the ground field of L. If L has more than 3 vertices, then the ground field of L is one of fields $\mathcal{F}L^4$. If L has 3 vertices, then the ground field of L is one of fields from $\mathcal{F}T$.

If $u = e_1 \cdot e_2 = 2$, then the ground field of Γ is equal to $\mathbb{Q}$.

Assume that $2 < u = e_1 \cdot e_2 < t$. Then the subset $\{e_1, e_2\}$ is hyperbolic and connected. Since Γ^{hyp} is connected, contains e_1, e_2 and has at least 4 vertices, obviously there exists a connected subgraph $\Gamma^{(4)}$ of Γ^{hyp} which contains e_1, e_2 and has four vertices. It is hyperbolic since it contains a hyperbolic subset $\{e_1,\ e_2\}$. Then $\Gamma^{(4)}$ is one of the hyperbolic graphs $\Gamma_i^{(4)}$, $i = 1, 2, 3, 4, 5$. Then the ground field of Γ is equal to one of fields $\Gamma_i^{(4)}(t)$, $1 \le i \le 5$.

This finishes the proof. $\square$

Degree of fields from $\mathcal{F}L^4$ is bounded by 2, and degree of fields from $\mathcal{F}T$ is bounded by 5.

For arithmetic hyperbolic reflection groups, the minimality $t = 14$ is especially important. Using the same methods as for the proof of Theorem 1 in [17], we can prove the following effective estimates.

Theorem 4. *The degree of fields from $\mathcal{F}\Gamma_1^{(4)}(14)$ is bounded ($\le$) by 24.*

The degree of fields from $\mathcal{F}\Gamma_2^{(4)}(14)$ is bounded by 39.

The degree of fields from $\mathcal{F}\Gamma_3^{(4)}(14)$ is bounded by 53.

The degree of fields from $\mathcal{F}\Gamma_4^{(4)}(14)$ is bounded by 120.

The degree of fields from $\mathcal{F}\Gamma_5^{(4)}(14)$ is bounded by 120.

Thus, the constant $N(14)$ of Theorem 1 can be taken to be $N(14) = 120$.

Proof. The proof requires long considerations and calculations. It will be given later in the special Sect. 5. $\square$

In the next section, we shall consider applications of these explicit estimates to arithmetic hyperbolic reflection groups.

4 Application to ground fields of arithmetic hyperbolic reflection groups

Let W be an arithmetic hyperbolic reflection group of dimension $n \ge 2$ and $\mathcal{M}$ its fundamental chamber. If $\mathcal{M}$ is not bounded, then the ground field of W is $\mathbb{Q}$ (see [27]). Since we are interested in possible ground fields of W, further we assume that $\mathcal{M}$ is bounded. Then any edge $r = v_1v_2$ of $\mathcal{M}$ defines a V-arithmetic finite edge chamber $\mathcal{M}(r)$ which is intersection of all half-spaces $\mathcal{H}_\delta^+$, $\delta \in P(\mathcal{M})$, such that the hyperplane $\mathcal{H}_\delta$ contains one of vertices v_1, v_2. The corresponding V-arithmetic edge graph is the Gram graph $\Gamma(r) = \Gamma(\mathcal{M}(r))$ of these elements $\delta \in P(\mathcal{M})$.

By [16] and [17], there exists $e \in P(\mathcal{M})$ which defines a narrow face $\mathcal{M}_e = \mathcal{H}_e \cap \mathcal{M}$ (or a face of minimality 14) of $\mathcal{M}$ (the same is valid for any hyperbolic closed convex polyhedron). It means that for the set

$$P(\mathcal{M}, e) = \{\delta \in P(\mathcal{M}) \mid \mathcal{H}_\delta \cap \mathcal{H}_e \ne \emptyset\}$$

of neighbouring to $\mathcal{M}_e$ faces of $\mathcal{M}$ one has

$$\delta_1 \cdot \delta_2 < 14 \ \ \forall\ \delta_1,\, \delta_2 \in P(\mathcal{M}, e).$$

By considering edges $r = v_1v_2$ in $\mathcal{M}_e$, we obtain many V-arithmetic edge chambers $\mathcal{M}(r)$ and V-arithmetic edge graphs of the minimality 14.

4.1 Ground fields of arithmetic hyperbolic reflection groups in dimensions $n \geq 10$

Let us assume that dimension $n \geq 10$. It was shown in [17], that the narrow face $\mathcal{M}_e$ contains an edge $r = v_1 v_2$ such that the corresponding edge chamber $\mathcal{M}(r)$ and its Gram graph $\Gamma = \Gamma(r)$ (they have minimality 14) have the hyperbolic connected component Γ^{hyp} with at least 4 vertices. By Theorem 1, then the degree of ground field of W is bounded by the constant $N(14)$. If we additionally apply Proposition 3, we obtain that the ground field of W belongs to one of finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$. By Theorem 4, the degree of this field is bounded by 120.

Here we want to give another proof of this result which permits to avoid enumeration of combinatorial types of 3-dimensional polyhedra with small number of vertices and excludes the most difficult and large set of fields $\mathcal{F}\Gamma_5^{(4)}(14)$ from the statement. This proof follows [19]. It additionally uses some important arguments by Vinberg from [30]. In [30], Vinberg has shown that for $n \geq 30$ there are no arithmetic hyperbolic reflection groups; for $n \geq 22$ their ground fields belong to the set $\{\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\cos \frac{2\pi}{7})\}$; for $n \geq 14$ their ground fields belong to $\mathcal{F}T$. Thus, Theorem 5 below can be viewed as some extension of these statements for $n \geq 10$.

Theorem 5. *In dimensions $n \geq 10$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_1^{(4)}(14)$, $\mathcal{F}\Gamma_2^{(4)}(14)$, $\mathcal{F}\Gamma_3^{(4)}(14)$ or $\mathcal{F}\Gamma_4^{(4)}(14)$. In particular, its degree is bounded ($\leq$) by 120.*

Proof. Let W be an arithmetic hyperbolic reflection group of dimension $n \geq 2$ and $\mathcal{M}$ its fundamental chamber with the Gram graph $\Gamma(P(\mathcal{M}))$. If $\mathcal{M}$ is not compact, then the ground field is $\mathbb{Q}$. Thus, further we can assume that $\mathcal{M}$ is compact. It is known, [27], that then $\mathcal{M}$ is a simple polyhedron which means that $\mathcal{M}$ is simplicial in its vertices: any vertex is contained in exactly n hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$.

Arguing like in the proof of Proposition 3, we obtain

Lemma 6. *Let W be an arithmetic hyperbolic reflection group, $\mathcal{M}$ its fundamental chamber and their ground field is different from fields from $\mathcal{F}L^4$ and $\mathcal{F}T$.*

Then any edge $r = v_1 v_2$ of $\mathcal{M}$ defines an edge polyhedron $\mathcal{M}(r)$ such that the corresponding vertices e_1 and e_2 of its Gram graph (that is the hyperplanes $\mathcal{H}_{e_1}$ and $\mathcal{H}_{e_2}$ contain only vertices v_1 and v_2 of the edge respectively) are joined by a broken edge (i.e. $u = e_1 \cdot e_2 > 2$).

For a vertex v of $\mathcal{M}$ we denote by $C(v)$ the Coxeter graph of v which is Gram graph of all $\delta \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_\delta$ contains v. Since $\mathcal{M}$ is compact, $C(v)$ has exactly n vertices.

Lemma 7. *Let W be an arithmetic hyperbolic reflection group, $\mathcal{M}$ its fundamental chamber and their ground field is different from fields from $\mathcal{F}L^4$, $\mathcal{F}T$*

and $\mathcal{F}\Gamma_i^{(4)}(14)$*,* $i = 1, 2, 3, 4$*. Let* $\mathcal{M}_e$*,* $e \in P(\mathcal{M})$*, be a narrow face of* $\mathcal{M}$ *of the minimality* 14.

Then for any vertex v *of* $\mathcal{M}$ *which is contained in* $\mathcal{M}_e$*, all connected components of the Coxeter graph* $C(v)$ *of* v *have one or two vertices only.*

Proof. Let e_1 be a vertex of $C(v)$ which is different from e. Let r be an edge of $\mathcal{M}$ defined by $C(v) - \{e_1\}$. It means that all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, which contain the edge r, belong to vertices of $C(v) - \{e_1\}$. Since $e \in C(v) - \{e_1\}$, the edge r belongs to $\mathcal{M}_e$. One of terminals of r is v, and $\mathcal{H}_{e_1}$ contains v, and it does not contain r. Let $v_2 \in \mathcal{M}_e$ be another terminal of r and $e_2 \in P(\mathcal{M})$ gives the hyperplane $\mathcal{H}_{e_2}$ which contains only v_2 and does not contain v. Thus, $\Gamma(r) = C(v) \cup \{e_2\}$ is the Gram graph of the edge polyhedron of $\mathcal{M}$ corresponding to $r = vv_2$. By Lemma 6, e_1, e_2 give the only broken edge of $\Gamma(r)$.

Since $r \subset \mathcal{M}_e$ and $\mathcal{M}_e$ has minimality 14, then $\Gamma(r)$ also has minimality 14.

Let us assume that the connected component of e_1 in Coxeter graph $C(v)$ has more than two vertices. Then there exists a connected subgraph Γ_{e_1} of $C(v)$ which has three vertices and contains e_1. Then $\Gamma_{e_1} \cup \{e_2\}$ is one of graphs $\Gamma_i^{(4)}$, $i = 1, 2, 3, 4, 5$, of minimality 14. It cannot be equal to $\Gamma_5^{(4)}$ because the connected components of e_1 in $\Gamma_5^{(4)} - \{e_2\}$ and e_2 in $\Gamma_5^{(4)} - \{e_1\}$ have two vertices. It follows that the ground field of $\mathcal{M}$ is one of fields $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4$. We obtain a contradiction.

Thus, any vertex $e_1 \neq e$ of the graph $C(v)$ has the connected component with one or two vertices. Obviously, the same will be valid for e as well.

This finishes the proof. □

The crucial topological argument of the proof of Theorem 5 (additional to Theorem 1 and existence of a narrow face of minimality 14) is as follows. By [17, Theorem 3.2.1], for any $0 \leq i \leq k$, $2 \leq k$ and $2k - 1 \leq m$, the average number $\alpha_m^{(i,k)}$ of i-dimensional faces in k-dimensional faces of any m-dimensional simple convex polyhedron satisfies the inequality

$$\alpha_m^{(i,k)} < \frac{C_{m-i}^{k-i}\left(C_{[m/2]}^i + C_{m-[m/2]}^i\right)}{C_{[m/2]}^k + C_{m-[m/2]}^k}. \tag{6}$$

In particular, for $n \geq 4$, the average number $\alpha_{n-1}^{(0,2)}$ of vertices of 2-dimensional faces of a narrow face $\mathcal{M}_e$ (of dimension $m = n - 1$) satisfies

$$\alpha_{n-1}^{(0,2)} < 4 + \begin{cases} \frac{4}{n-2} & \text{if} \quad n \text{ is even}, \\ \frac{4}{n-3} & \text{if} \quad n \text{ is odd}. \end{cases} \tag{7}$$

Now let us assume that the ground field is different from the fields from the sets $\mathcal{F}L$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4$. Following Vinberg [29], let us estimate the number A of non-right (i.e $\neq \pi/2$) 2-dimensional angles of $\mathcal{M}_e$.

Let v be a vertex of $\mathcal{M}_e$ and $C(v)$ the Coxeter graph of v in $\mathcal{M}$. Any 2-dimensional angle of $\mathcal{M}_e$ with the vertex v is defined by a subset of two distinct

vertices $\{\delta_1, \delta_2\} \subset C(v) - \{e\}$. Really, $C(v) - \{\delta_1, \delta_2\}$ define perpendicular vectors to hyperplanes containing the plane of the angle, and $C(v) - \{\delta_1\}$, $C(v) - \{\delta_2\}$ similarly define edges of the angle. It is easy to see that the angle is not right if and only if δ_1 and δ_2 belong to one connected component of the graph $C(v)$. Thus, the number A_v of non-right 2-dimensional angles of $\mathcal{M}_e$ with the vertex v is equal to the number of subsets of two distinct vertices of $C(v) - \{e\}$ which belong to one connected component of the graph $C(v)$. By Lemma 7, $A_v \leq [(n-1)/2]$. Thus,

$$\left[\frac{n-1}{2}\right]\alpha_0 \ \geq \ A \tag{8}$$

where α_0 is the number of vertices of $\mathcal{M}_e$.

Each hyperbolic triangle has at least two non-right angles. Each hyperbolic quadrangle has at least one non-right angle. Denoting by α_2^l the number of 2-dimensional faces of $\mathcal{M}_e$ with l vertices and by α_2 the number of all 2-dimensional faces of $\mathcal{M}_e$, we obtain

$$A \geq \sum_{l\geq 3}(5-l)\alpha_2^l = 5\alpha_2 - \sum_{l\geq 3} l\alpha_2^l = 5\alpha_2 - \alpha_{n-1}^{(0,2)}\alpha_2 = (5-\alpha_{n-1}^{(0,2)})\alpha_2. \tag{9}$$

Since $\mathcal{M}_e$ is a simple and $(n-1)$-dimensional convex polyhedron, we have

$$\frac{\alpha_0(n-1)(n-2)}{2} = \alpha_2\alpha_{n-1}^{(0,2)}. \tag{10}$$

From (8), (9) and (10), we obtain

$$\alpha_{n-1}^{(0,2)}\left(\frac{(n-1)(n-2)}{2} + \left[\frac{n-1}{2}\right]\right) \geq \frac{5(n-1)(n-2)}{2}\ .$$

From (7), we get

$$\left(4+\frac{4}{n-2}\right)\left(\frac{(n-1)(n-2)}{2}+\frac{n-2}{2}\right) > \frac{5(n-1)(n-2)}{2}$$

for even n, and

$$\left(4+\frac{4}{n-3}\right)\left(\frac{(n-1)(n-2)}{2}+\frac{n-1}{2}\right) > \frac{5(n-1)(n-2)}{2}$$

for odd n. It follows $n \leq 9$ which contradicts to the assumption $n \geq 10$.

This finishes the proof. $\square$

4.2 Ground fields of arithmetic hyperbolic reflection groups in dimensions $n \geq 6$

Let us introduce one more set of fields. Let us consider plane (or Fuchsian) arithmetic hyperbolic reflection groups W with a quadrangle fundamental polygon K of minimality 14. We remind that this means that $P(K) = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ satisfies the condition

$$\delta_i \cdot \delta_j < 14,\ \forall\ \delta_i, \delta_j \in P(K).$$

Respectively, we call K as *arithmetic quadrangle of minimality* 14.

Definition 8. *We denote by* $\Gamma_{2,4}(14)$ *the set of Gram graphs* $\Gamma(P(K))$ *of all arithmetic quadrangles* K *of minimality* 14*. The set* $\mathcal{F}_{2,4}(14)$ *consists of all their ground fields.*

By Borel [4] and Takeuchi [25], for fixed $g \geq 0$ and $t \geq 0$, the number of arithmetic Fuchsian groups with signatures $(g;\ e_1,\, e_2,\, \dots,\, e_t)$ is finite. Applying this result to $g = 0$ and $t = 4$, we obtain that sets of arithmetic quadrangles $\Gamma_{2,4}$ and their ground fields $\mathcal{F}_{2,4}$ are finite. Then their subsets $\Gamma_{2,4}(14)$ and $\mathcal{F}_{2,4}(14)$ are also finite.

Moreover, in [25, pages 383–384] an upper bound n_0 of the degree of ground fields of Fuchsian groups with signatures $(g;\ e_1,\, e_2,\, \dots,\, e_t)$ is given. It is

$$n_0 = (b + \log_e C(g,t))/\log_e(a/(2\pi)^{4/3}) \tag{11}$$

where

$$a = 29.099,\ b = 8.3185,\ C(g,t) = 2^{2g+t-2}(2g+t-2)^{2/3}$$

(here a and b are due to Odlyzko). It follows that

$$[\mathbb{K} : \mathbb{Q}] \leq 11 \ \text{ for } \ \mathbb{K} \in \mathcal{F}_{2,4} \supset \mathcal{F}_{2,4}(14). \tag{12}$$

We have the following main result of the paper.

Theorem 9. *In dimensions* $n \geq 6$*, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields* $\mathcal{F}L^4$*,* $\mathcal{F}T$*,* $\mathcal{F}\Gamma_i^{(4)}(14)$*,* $1 \leq i \leq 5$*, and* $\mathcal{F}\Gamma_{2,4}(14)$*. In particular, its degree is bounded by* 120*.*

Proof. We use notations of Sect. 4.1 above. By (7), for $n \geq 6$ a narrow face $\mathcal{M}_e$ has $\alpha_{n-1}^{(0,2)} < 5$. Thus, $\mathcal{M}_e$ has a triangle or quadrangle (2-dimensional) face. Let us consider both cases.

By Lemma 6, we have

Lemma 10. *Let* W *be an arithmetic hyperbolic reflection group,* $\mathcal{M}$ *its fundamental chamber and their ground field is different from fields from* $\mathcal{F}L^4$ *and* $\mathcal{F}T$*.*

Then $\mathcal{M}$ *has no triangle faces (2-dimensional).*

Proof. Assume $\mathcal{M}$ contains a triangle 2-dimensional face. Then the edge polyhedron of $\mathcal{M}$ corresponding to the edge v_1v_2 of two vertices v_1 and v_2 of this triangle has the corresponding elements e_1 and e_2 (from Lemma 6) such that the hyperplanes $\mathcal{H}_{e_1}$ and $\mathcal{H}_{e_2}$ have a common point which is the third vertex of the triangle. Then $e_1 \cdot e_2 \leq 2$. This contradicts Lemma 6.

This finishes the proof. ∎

Let $\mathcal{M}_e$ has a triangle face. By Lemma 10, then the ground field of $\mathcal{M}$ belongs to $\mathcal{F}L^4$ or $\mathcal{F}T$ as required.

Lemma 11. *Let W be an arithmetic hyperbolic reflection group, $\mathcal{M}$ its fundamental chamber and their ground field is different from fields from $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}$, $1 \le i \le 5$.*

Let $\mathcal{M}_e$, $e \in P(\mathcal{M})$, be a narrow face of $\mathcal{M}$ of minimality 14. Let $\mathcal{M}_4$ be a quadrangle face of $\mathcal{M}_e$. Let $Q \subset P(\mathcal{M})$ consists of all $n-2$ elements which are perpendicular to the plane of $\mathcal{M}_4$, and $\delta_j \in P(\mathcal{M})$, $j = 1, 2, 3, 4$, are additional 4 elements which are perpendicular to four edges of $\mathcal{M}_4$.

Then all elements δ_j are perpendicular to Q and $\mathcal{M}_4$ is an arithmetic quadrangle of minimality 14 with $P(\mathcal{M}) = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ and with the same ground field as $\mathcal{M}$. Thus, the ground field of W belongs to $\mathcal{F}_{2,4}(14)$.

Proof. We assume that δ_1, δ_2, δ_3 and δ_4 are perpendicular to four consecutive edges of $\mathcal{M}_4$. The quadrangle $\mathcal{M}_4$ has a non-right angle. We can assume that δ_1 and δ_2 are perpendicular to edges of this angle. Since the angle is non-right, δ_1 and δ_2 belong to one connected component of the Coxeter graph of the vertex of the angle. By Lemma 7, then δ_1 and δ_2 give a connected component of the graph. It follows that δ_1 and δ_2 are perpendicular to Q.

Assume that δ_3 is not perpendicular to Q and $\delta_3 \cdot e > 0$ for $e \in Q$. Then $\delta_1, \delta_2, \delta_3, e$ define an edge graph $\Gamma_4^{(4)}$ (if $\delta_2 \cdot \delta_3 > 0$) or $\Gamma_5^{(4)}$ (if $\delta_2 \cdot \delta_3 = 0$) of minimality 14. It follows that the ground field of $\mathcal{M}$ belongs to $\mathcal{F}\Gamma_4^{(4)}(14)$ or $\mathcal{F}\Gamma_5^{(4)}(14)$, and we get a contradiction. Thus, δ_3 is perpendicular to Q. Similarly we can prove that δ_4 is perpendicular to Q.

This finishes the proof of the lemma. $\square$

If $\mathcal{M}_e$ contains a quadrangle, by Lemma 11, the ground field of W belongs to $\mathcal{F}_{2,4}(14)$ as required. This finishes the proof of the theorem. $\square$

4.3 Ground fields of arithmetic hyperbolic reflection groups in dimensions $n \ge 4$

Unfortunately, in dimension $n \ge 4$ we don't know similar results to Theorems 5 and 9. Possibly, the recent preprint by Agol, Belolipetsky, Storm and Whyte [2] contains some similar information. It gives some effective bounds on degrees and discriminants of ground fields of arithmetic hyperbolic reflection groups for $n \ge 4$. Unfortunately, they are not explicit, it seems.

On the other hand, the following result had been obtained in [20].

Theorem 12. *([20]) For $n \ge 4$, the ground field of any n-dimensional arithmetic hyperbolic reflection group is either the ground field of one of $n-1$ or $n-2$-dimensional arithmetic hyperbolic reflection group with a fundamental chamber of minimality 14, or a field from $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4$.*

In particular, its degree is bounded by the maximum of degrees of ground fields of $n-1$ and $n-2$-dimensional arithmetic hyperbolic reflection groups with a fundamental chamber of minimality 14, and of 120 (according to Theorem 4 of this paper).

Proof. We repeat arguments of [20].

Let $\mathcal{M}$ be a fundamental chamber of W. We can assume that $\mathcal{M}$ is compact and the ground field of $\mathcal{M}$ is not contained in $\mathcal{F}L^4$ and $\mathcal{F}T$. Let $\mathcal{M}_e$, $e \in P(\mathcal{M})$, be a face of $\mathcal{M}$ of minimality 14.

If all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M}, e) - \{e\}$, are perpendicular to $\mathcal{M}_e$ (equivalently, $\delta \cdot e = 0$), then $\mathcal{M}_e$ is a fundamental chamber of $n-1$-dimensional arithmetic hyperbolic reflection group with the same ground field as $\mathcal{M}$. Obviously, $P(\mathcal{M}_e) = P(\mathcal{M}, e) - \{e\}$, and $\mathcal{M}_e$ has minimality 14.

If this is not the case, there exists $f \in P(\mathcal{M}, e) - \{e\}$ such that $f \cdot e > 0$ (equivalently, f and e are connected by a thin edge in Gram graph of $\mathcal{M}$). Then $\mathcal{M}_{e,f} = \mathcal{M} \cap \mathcal{H}_e \cap \mathcal{H}_f$ is $n-2$-dimensional face of $\mathcal{M}$. Let $P(\mathcal{M}, e, f)$ be the set of all $\delta \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_\delta$ intersects the codimension-two subspace $\mathcal{H}_e \cap \mathcal{H}_f$ (then $\mathcal{M} \cap \mathcal{H}_e \cap \mathcal{H}_f \cap \mathcal{H}_\delta$ is a codimension-three face of $\mathcal{M}$ if δ is different from e and f). If $\mathcal{H}_\delta \perp \mathcal{H}_e \cap \mathcal{H}_f$ (equivalently, $\delta \cdot e = \delta \cdot f = 0$) for all $\delta \in P(\mathcal{M}, e, f) - \{e, f\}$, then $\mathcal{M}_{e,f}$ is a fundamental chamber of an arithmetic hyperbolic reflection group of dimension $n-2$ with the same ground field as $\mathcal{M}$. Obviously, $P(\mathcal{M}_{e,f}) = P(\mathcal{M}, e, f) - \{e, f\}$, and $\mathcal{M}_{e,f}$ has minimality 14.

If this is not the case, there is $g \in P(\mathcal{M}, e, f) - \{e, f\}$ such that $\mathcal{H}_g$ is not perpendicular to $\mathcal{H}_e \cap \mathcal{H}_f$. This means that either $g \cdot e > 0$ or $g \cdot f > 0$. Thus, the Gram graph of e, f, g is a connected negative definite (i. e. connected Coxeter) graph.

We consider an edge r in the face $\mathcal{M}_{e,f} = \mathcal{M} \cap \mathcal{H}_e \cap \mathcal{H}_f$ of $\mathcal{M}$ such that r terminates in the hyperplane $\mathcal{H}_g$. Thus one of vertices of r is contained in $\mathcal{H}_g$ while the other is not (equivalently, r is not contained in $\mathcal{H}_g$). The existence of such an edge is obvious. Let $h \in P(\mathcal{M})$ defines the hyperplane $\mathcal{H}_h$ which contains only the vertex of r which does not belong to $\mathcal{H}_g$. Then g and h are joined by a broken edge in the edge graph $\Gamma(r)$ of r (here we can additionally assume that the ground field does not belong to $\mathcal{F}L^4$ and $\mathcal{F}T$). The four elements $\{e, f, g, h\}$ define a connected hyperbolic subgraph of $\Gamma(r)$ with four vertices. It is one of graphs $\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4$. Then the ground field of $\mathcal{M}$ belongs to one of sets $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 4$. This finishes the proof. $\square$

Theorem 12 shows that ground fields of arithmetic hyperbolic reflection groups which are different from fields of $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4$, come up from 2-dimensional and 3-dimensional arithmetic hyperbolic reflection groups.

4.4 Ground fields of arithmetic hyperbolic reflection groups in dimension $n = 3$

Finiteness of the number of maximal arithmetic hyperbolic reflection groups of dimension $n = 3$ was proved by Agol in [1]. It follows that the number of ground fields of 3-dimensional arithmetic hyperbolic reflection groups is also finite. Unfortunately, an explicit bound of degrees of these fields is not known (by the author's knowledge). We remind that by [16], the set of ground fields of

arithmetic hyperbolic reflection groups of the fixed degree is finite and can be effectively found. Thus, an explicit bound of the degree is the crucial problem in finding of ground fields of 3-dimensional hyperbolic reflection groups.

4.5 Ground fields of arithmetic hyperbolic reflection groups in dimension $n = 2$

Finiteness of the number of maximal arithmetic hyperbolic reflection groups of dimension $n = 2$ was proved by Long, Maclachlan and Reid [15]. They proved finiteness of maximal arithmetic Fuchsian groups of genus 0. Their ground fields contain ground fields of all arithmetic hyperbolic reflection groups.

Let Γ be a cocompact maximal arithmetic Fuchsian group of genus 0. Using results by M.-F. Vigneras [26] and Zograf [33], the bound

$$\mathrm{Area}\left(\mathbb{H}^2/\Gamma\right) \leq \frac{128\pi}{3} \tag{13}$$

of the area of the arithmetic quotient was obtained in [15]. Let Γ has signature $(0; e_1, \dots, e_t)$ where $e_i \geq 2$. Then the area of the quotient is equal to

$$\mathrm{Area}\left(\mathbb{H}^2/\Gamma\right) = 2\pi\left(t - 2 - \sum_{i=1}^{t}\frac{1}{e_i}\right).$$

Since $e_i \geq 2$, by (13), we obtain $2\pi(t - 2 - t/2) \leq 128\pi/3$, and $t \leq 46$. By the result of Takeuchi (11), we obtain $n_0 \leq 44$.

Thus,*degree of the ground field of any arithmetic Fuchsian group of genus 0 is less or equal to* 44. In particular, *degree of the ground field of any 2-dimensional arithmetic hyperbolic reflection group is less or equal to* 44.

Summarising above results, we see that an explicit bound of degrees of ground fields of arithmetic hyperbolic reflection groups remains unknown in dimensions $n = 3$, 4, 5 only. Moreover, the dimension $n = 3$ is crucial for this problem. If one finds this bound for $n = 3$, we will know it for all remaining dimensions $n = 4$ and $n = 5$.

5 Ground fields of V-arithmetic connected finite edge graphs with four vertices of the minimality 14.

Here we shall obtain explicit upper bounds of degrees of fields from the finite sets $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$ (see Definition 2), and prove Theorem 4. Moreover, our considerations will deliver important information about these sets of fields.

Like for the proof of Theorem 1 from [17], we use the following general results from [17] (we use corrections from Section 7).

Theorem 13. *([17, Theorem 1.2.1]) Let $\mathbb{F}$ be a totally real algebraic number field, and let each embedding $\sigma : \mathbb{F} \to \mathbb{R}$ corresponds to an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$ where*

$$\prod_\sigma \frac{b_\sigma - a_\sigma}{4} < 1.$$

In addition, let the natural number m and the intervals $[s_1, t_1], \dots, [s_m, t_m]$ in $\mathbb{R}$ be fixed. Then there exists a constant $N(s_i, t_i)$ such that, if α is a totally real algebraic integer and if the following inequalities hold for the embeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \le \tau(\alpha) \le t_i \quad for \quad \tau = \tau_1, \dots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \le \tau(\alpha) \le b_{\tau|\mathbb{F}} \quad for \quad \tau \ne \tau_1, \dots, \tau_m,$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \le N(s_i, t_i).$$

Theorem 14. *([17, Theorem 1.2.2]) Under the conditions of Theorem 13, $N(s_i, t_i)$ can be taken to be $N(s_i, t_i) = N$, where N is the least natural number solution of the inequality*

$$N \ln(1/R) - M \ln(2N + 2) - \ln B \ge \ln S. \tag{14}$$

Here

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = \sqrt{|\mathrm{discr}\ \mathbb{F}|}; \tag{15}$$

$$R = \sqrt{\prod_\sigma \frac{b_\sigma - a_\sigma}{4}}, \quad S = \prod_{i=1}^m \frac{2er_i}{b_{\sigma_i} - a_{\sigma_i}} \tag{16}$$

where

$$\sigma_i = \tau_i|\mathbb{F}, \quad r_i = \max\{|b_i - a_{\sigma_i}|, |b_{\sigma_i} - a_i|\}. \tag{17}$$

We note that the proof of Theorems 13 and 14 uses a variant of Fekete's Theorem (1923) about existence of non-zero integer polynomials of bounded degree which differ only slightly from zero on appropriate intervals. See [17, Theorem 1.1.1] (see its corrections in Section 7, Theorems 16, 17).

5.1 Fields from $\mathcal{F}\Gamma_1^{(4)}(14)$

For $\Gamma_1^{(4)}(14)$ (see Figure 4) we assume that integers $s, k, r, p \ge 3$. Subgraphs $\Gamma_1^{(4)} - \{e_1\}$ and $\Gamma_1^{(4)} - \{e_2\}$ must be Coxeter graphs. It follows that we must consider only (up to obvious symmetries) the following cases: either $s = k = 3$ and $5 \ge r \ge p \ge 3$, or $s = p = 3$ and $5 \ge r \ge k \ge 4$; the totally real algebraic integer u satisfies the inequality $2 < u < 14$.

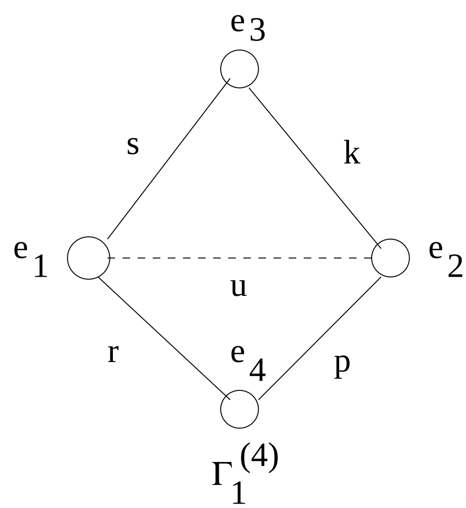

Figure 4: The graph $\Gamma_1^{(4)}$

The corresponding Gram matrix is

$$\left(\begin{array}{cccc} -2 & u & 2\cos\frac{\pi}{s} & 2\cos\frac{\pi}{r} \\ u & -2 & 2\cos\frac{\pi}{k} & 2\cos\frac{\pi}{p} \\ 2\cos\frac{\pi}{s} & 2\cos\frac{\pi}{k} & -2 & 0 \\ 2\cos\frac{\pi}{r} & 2\cos\frac{\pi}{p} & 0 & -2 \end{array}\right), \tag{18}$$

where all entries are algebraic integers, and its determinant $d(u)$ is given by the equality

$$-\frac{d(u)}{4} = \left(u + 2\left(\cos\frac{\pi}{r}\cos\frac{\pi}{p} + \cos\frac{\pi}{k}\cos\frac{\pi}{s}\right)\right)^2 - \left(\cos\frac{2\pi}{k} + \cos\frac{2\pi}{p}\right)\left(\cos\frac{2\pi}{r} + \cos\frac{2\pi}{s}\right). \tag{19}$$

The ground field $\mathbb{K}$ is generated by the cyclic products

$$\mathbb{K} = (u^2, u\cos(\pi/s)\cos(\pi/k), u\cos(\pi/p)\cos(\pi/r), \cos^2(\pi/s), \cos^2(\pi/k),$$
$$\cos^2(\pi/p), \cos^2(\pi/r), \cos(\pi/s)\cos(\pi/k)\cos(\pi/p)\cos(\pi/r)).$$

Here and in what follows we always denote by $\sigma^{(+)} : \mathbb{K} \to \mathbb{R}$ the geometric (the identity) embedding, and by $\sigma : \mathbb{K} \to \mathbb{R}$ all other embeddings $\sigma \neq \sigma^{(+)}$. We have $\sigma^{(+)}(d(u)) < 0$ because $\Gamma_1^{(4)}$ is hyperbolic for $\sigma^{(+)}$ since e_1, e_2 define a hyperbolic subgraph because $u = e_1 \cdot e_2 > 2$. And $\sigma(d(u)) > 0$ since $\Gamma_1^{(4)}$ is negative definite for σ. In particular, $-2 < \sigma(u) < 2$ by Cauchy inequality. Thus, $\Gamma_1^{(4)}$ is V-arithmetic if and only if

$$\sigma\left(u + 2(\cos\frac{\pi}{r}\cos\frac{\pi}{p} + \cos\frac{\pi}{k}\cos\frac{\pi}{s})\right)^2 <$$

$$\sigma\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right). \tag{20}$$

for each $\sigma \neq \sigma^{(+)}$.

We have $\mathbb{K}=\mathbb{Q}(u^2)$ since $4<\sigma^{(+)}(u^2)<14^2$ and $0<\sigma(u^2)<4$ if $\sigma\neq\sigma^{(+)}$. We have $[\mathbb{K}(u):\mathbb{K}]=2$ if $u\notin\mathbb{K}$. If $\tau:\mathbb{K}(u)\to\mathbb{R}$ gives $\tau|\mathbb{K}=\sigma^{(+)}$, then $\tau(u)=\pm u$ (where u is taken for the geometric embedding $\sigma^{(+)}$), and either $2<\tau(u)<14$ or $-14<\tau(u)<-2$. The last inequality is possible, only if u does not belong to $\mathbb{K}$.

If $\tau|\mathbb{K}=\sigma\neq\sigma^{(+)}$, then by (5.1),

$$-2\widetilde{\tau}\left(\cos\frac{\pi}{r}\cos\frac{\pi}{p}+\cos\frac{\pi}{k}\cos\frac{\pi}{s}\right)-$$

$$-\sqrt{\tau\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right)}.$$

$$<\tau(u)<-2\widetilde{\tau}\left(\cos\frac{\pi}{r}\cos\frac{\pi}{p}+\cos\frac{\pi}{k}\cos\frac{\pi}{s}\right)+$$

$$\sqrt{\tau\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right)}$$

where $\tilde{\tau}$ extends τ. Thus, $\tau(u)$ belongs to the interval of the length

$$2\sqrt{\tau\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right)}.$$

We apply Theorems 13 and 14 to

$$\mathbb{F}=\mathbb{Q}(\cos^2(\pi/s),\cos^2(\pi/k),\cos^2(\pi/p),\cos^2(\pi/r))$$

and $\alpha=u$ to bound $[\mathbb{F}(u):\mathbb{F}]$. Since $\mathbb{K}=\mathbb{F}(u^2)$, then $m=[\mathbb{F}(u):\mathbb{K}]\leq 2$. From considerations above, we have

$$M=[\mathbb{F}:\mathbb{Q}],\quad B=\sqrt{|\mathrm{discr}\ \mathbb{F}|},$$

$$R=\frac{N_{\mathbb{F}/\mathbb{Q}}\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right)^{1/4}}{2^{M/2}},$$

$$S=\left(\frac{16\cdot e}{\sqrt{\left((\cos\frac{2\pi}{k}+\cos\frac{2\pi}{p})(\cos\frac{2\pi}{r}+\cos\frac{2\pi}{s})\right)}}\right)^m,$$

$$[\mathbb{K}:\mathbb{Q}]\leq NM/m$$

where N is the least natural solution of the inequality

$$N\ln(1/R)-M\ln(2N+2)-\ln B\geq\ln S.$$

By direct calculations, we obtain the following.

Let $k = s = 3$. Then $m = 1$.

For $r = p = 3$, we obtain $M = 1$, $B = 1$, $R = 1/\sqrt{2}$, $S = 16e$. Then $N = 22$ and $[\mathbb{K} : \mathbb{Q}] \leq 22$.

For $r = 4$, $p = 3$, we obtain $M = 1$, $B = 1$, $R = 1/2^{3/4}$, $S = 16e\sqrt{2}$. Then $N = 15$ and $[\mathbb{K} : \mathbb{Q}] \leq 15$.

For $r = 5$, $p = 3$, we obtain $M = 2$, $B = \sqrt{5}$, $R = 1/2^{3/2}$, $S = 16e/\sqrt{1/2 - cos(2\pi/5)}$. Then $N = 12$ and $[\mathbb{K} : \mathbb{Q}] \leq 24$.

For $r = p = 4$, we obtain $M = 1$, $B = 1$, $R = 1/2$, $S = 32e$. Then $N = 12$ and $[\mathbb{K} : \mathbb{Q}] \leq 12$.

For $r = 5$, $p = 4$, we obtain $M = 2$, $B = \sqrt{5}$, $R = 1/4$, $S = 32e/\sqrt{1 - 2cos(2\pi/5)}$. Then $N = 9$ and $[\mathbb{K} : \mathbb{Q}] \leq 18$.

For $r = p = 5$, we obtain $M = 2$, $B = \sqrt{5}$, $R = 1/4$, $S = 16e/(1/2 - cos(2\pi/5))$. Then $N = 9$ and $[\mathbb{K} : \mathbb{Q}] \leq 18$.

Now let $s = p = 3$. Then $m = 1$ or $m = 2$.

For $r = k = 4$, we obtain $M = 1$, $B = 1$, $R = 1/2$, $S = (32e)^m$. Then $N = 12$ and $[\mathbb{K} : \mathbb{Q}] \leq 12$ for $m = 1$; $N = 19$ and $[\mathbb{K} : \mathbb{Q}] \leq 9$ for $m = 2$.

For $r = 5$, $k = 4$, we obtain $m = 1$, $M = 2$, $B = \sqrt{5}$, $R = 1/4$, $S = 32e/\sqrt{1 - 2cos(2\pi/5)}$. Then $N = 9$ and $[\mathbb{K} : \mathbb{Q}] \leq 18$.

For $r = k = 5$, we obtain $M = 2$, $B = \sqrt{5}$, $R = 1/4$, $S = (16e/(1/2 - cos(2\pi/5))^m$. Then $N = 9$ and $[\mathbb{K} : \mathbb{Q}] \leq 18$ for $m = 1$, and $N = 14$ and $[\mathbb{K} : \mathbb{Q}] \leq 14$ for $m = 2$.

Note that the bound for $[\mathbb{K} : \mathbb{Q}]$ is always worse for $m = 1$ than for $m = 2$. It follows from our method. Further, in similar considerations, we can consider $m = 1$ only.

Thus, our upper bound for degrees of fields from $\mathcal{F}\Gamma_1^{(4)}(14)$ is 24.

5.2 Fields from $\mathcal{F}\Gamma_2^{(4)}(14)$

For $\Gamma_2^{(4)}(14)$ (see Figure 5), $s, k, p \geq 3$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following possibilities: $3 \leq s \leq k \leq 5$, $p = 3$; $s = k = 3$, $p = 4, 5$.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2 \frac{\pi}{s},\ \cos^2 \frac{\pi}{k},\ \cos^2 \frac{\pi}{p},\ u^2,\ u \cos \frac{\pi}{s} \cos \frac{\pi}{k}\,.$$

This case is similar to $\mathcal{F}\Gamma_1^{(4)}(14)$. The determinant $d(u)$ of the Gram matrix is determined by the equality

$$-\frac{d(u)}{4} = \sin^2 \frac{\pi}{p}\, u^2 + 4 \cos \frac{\pi}{s} \cos \frac{\pi}{k}\, u + 4(\cos^2 \frac{\pi}{s} + \cos^2 \frac{\pi}{k} + \cos^2 \frac{\pi}{p} - 1).$$

Let

$$D = 16 \cos^2 \frac{\pi}{s} \cos^2 \frac{\pi}{k} + 16 \sin^2 \frac{\pi}{p}(1 - \cos^2 \frac{\pi}{s} - \cos^2 \frac{\pi}{k} - \cos^2 \frac{\pi}{p})$$

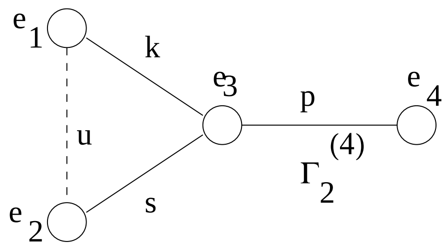

Figure 5: The graph $\Gamma_2^{(4)}$

be the discriminant of this quadratic polynomial of the variable u. The graph $\Gamma_4^{(4)}(14)$ is V-arithmetic if and only if for $\tau : \mathbb{K}(u) \to \mathbb{R}$ which is different from $\sigma^{(+)}$ on $\mathbb{K}$, one has

$$\frac{-4\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\right) - \sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{p})} < \tau(u) <$$

$$< \frac{-4\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\right) + \sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{p})}$$

where $\widetilde{\tau}$ extends τ.

Thus, $\tau(u)$ belongs to an interval of the length $2\sqrt{\tau\left(D/(4\sin^4\frac{\pi}{p})\right)}$.

We can apply Theorems 13, 14 to $\mathbb{F} = \mathbb{Q}(\cos^2\frac{\pi}{s},\ \cos^2\frac{\pi}{k},\ \cos^2\frac{\pi}{p})$ and $\alpha = u$. Then

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = \sqrt{|\text{discr }\mathbb{F}|},$$

$$R = \frac{N_{\mathbb{F}/\mathbb{Q}}(D)^{1/4}}{N_{\mathbb{F}/\mathbb{Q}}\left(\sin^2(\pi/p)\right)^{1/2}2^M}\ , \quad S = \frac{32e\sin^2(\pi/p)}{\sqrt{D}},$$

$[\mathbb{K} : \mathbb{Q}] \le NM$ where N is the smallest natural solution of the inequality

$$N\ln(1/R) - \ln(2N+2) - \ln B \ge \ln S.$$

We obtain:
if $s = k = p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 39$; if $s = 3$, $k = 4$, $p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 21$;
if $s = 3$, $k = 5$, $p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 34$; if $s = k = 4$, $p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 14$;
if $s = 4$, $k = 5$, $p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 22$; if $s = k = 5$, $p = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 24$;
if $s = k = 3$, $p = 4$, then $[\mathbb{K} : \mathbb{Q}] \le 22$; if $s = k = 3$, $p = 5$, then $[\mathbb{K} : \mathbb{Q}] \le 32$.

Thus, our upper bound for degrees of fields from $\mathcal{F}\Gamma_2^{(4)}(14)$ is 39.

5.3 Fields from $\mathcal{F}\Gamma_3^{(4)}(14)$

For $\Gamma_3^{(4)}(14)$ (see Figure 6), $s \ge 2$, $k, r \ge 3$ are natural numbers, and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following

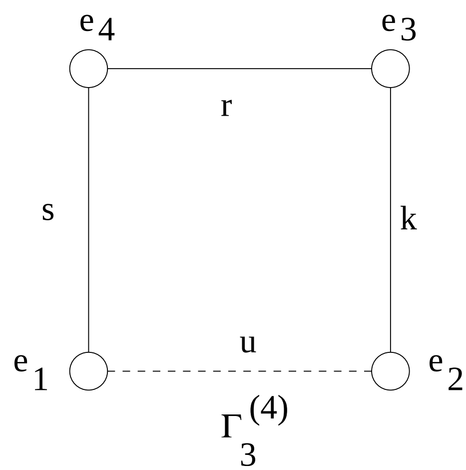

Figure 6: The graph $\Gamma_3^{(4)}$

possibilities: $s = 2$, $k = 3$, $r = 3, 4, 5$; $s = 2$, $k = 4, 5$, $r = 3$; $3 \le s \le k \le 5$, $r = 3$; $s = k = 3$, $r = 4, 5$.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2\frac{\pi}{s},\ \cos^2\frac{\pi}{k},\ \cos^2\frac{\pi}{r},\ u^2,\ u\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\,.$$

This case is similar to $\mathcal{F}\Gamma_1^{(4)}(14)$ and $\mathcal{F}\Gamma_2^{(4)}(14)$. The determinant $d(u)$ of the Gram matrix is determined by the equality

$$-\frac{d(u)}{4} = \sin^2\frac{\pi}{r}\,u^2 + 2\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\,u + 4\cos^2\frac{\pi}{r} - 4\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k}.$$

Let

$$D = 4\cos^2\frac{\pi}{s}\cos^2\frac{\pi}{k}\cos^2\frac{\pi}{r} + 16\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k}\sin^2\frac{\pi}{r} - 16\sin^2\frac{\pi}{r}\cos^2\frac{\pi}{r}$$

be the discriminant of this quadratic polynomial of the variable u. The graph $\Gamma_3^{(4)}(14)$ is V-arithmetic if and only if for $\tau : \mathbb{K}(u) \to \mathbb{R}$ which is different from $\sigma^{(+)}$ on $\mathbb{K}$, one has

$$\frac{-2\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\right) - \sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{r})} < \tau(u) <$$

$$< \frac{-2\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\right) + \sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{r})}$$

where $\widetilde{\tau}$ extends τ.

Thus, $\tau(u)$ belongs to an interval of the length $2\sqrt{\tau\left(D/(4\sin^4\frac{\pi}{r})\right)}$

We can apply Theorems 13, 14 to $\mathbb{F} = \mathbb{Q}(\cos^2 \frac{\pi}{s},\ \cos^2 \frac{\pi}{k},\ \cos^2 \frac{\pi}{r})$ and $\alpha = u$. Then

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = \sqrt{|\mathrm{discr}\ \mathbb{F}|},$$

$$R = \frac{N_{\mathbb{F}/\mathbb{Q}}(D)^{1/4}}{N_{\mathbb{F}/\mathbb{Q}}\left(\sin^2(\pi/r)\right)^{1/2} 2^M}\ , \quad S = \frac{32e\sin^2(\pi/r)}{\sqrt{D}},$$

$[\mathbb{K} : \mathbb{Q}] \le NM$ where N is the smallest natural solution of the inequality

$$N \ln(1/R) - M\ln(2N+2) - \ln B \ge \ln S.$$

We obtain:
if $s = 2$, $k = r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 83$; if $s = 2$, $k = 3$, $r = 4$, then $[\mathbb{K} : \mathbb{Q}] \le 45$; if $s = 2$, $k = 3$, $r = 5$, then $[\mathbb{K} : \mathbb{Q}] \le 66$; if $s = 2$, $k = 4$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 28$;
if $s = 2$, $k = 5$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 48$; if $s = k = r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 37$;
if $s = r = 3$, $k = 4$, then $[\mathbb{K} : \mathbb{Q}] \le 18$; if $s = r = 3$, $k = 5$, then $[\mathbb{K} : \mathbb{Q}] \le 24$;
if $s = 4$, $k = 4$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 9$; if $s = 4$, $k = 5$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] = 2$;
if $s = k = 5$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] = 2$; if $s = k = 3$, $r = 4$, then $[\mathbb{K} : \mathbb{Q}] \le 17$;
if $s = k = 3$, $r = 5$, then $[\mathbb{K} : \mathbb{Q}] = 2$.

For $s = 2$ we can improve these estimates considering $\alpha = u^2$. In this case, $0 < \tau(u^2) < \tau\left(D/(4\sin^4(\pi/r))\right)$, and we can apply Theorems 13, 14 to the same $\mathbb{F}$ and

$$R = \frac{N_{\mathbb{F}/\mathbb{Q}}(D)^{1/2}}{N_{\mathbb{F}/\mathbb{Q}}(\sin^2(\pi/r))4^M}, \quad S = \frac{2 \cdot e \cdot 14^2 \cdot 4 \cdot \sin^4(\pi/r)}{D}\ .$$

We obtain:
if $s = 2$, $k = r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 53$; if $s = 2$, $k = 3$, $r = 4$, then $[\mathbb{K} : \mathbb{Q}] \le 31$; if $s = 2$, $k = 3$, $r = 5$, then $[\mathbb{K} : \mathbb{Q}] \le 32$; if $s = 2$, $k = 4$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 19$; if $s = 2$, $k = 5$, $r = 3$, then $[\mathbb{K} : \mathbb{Q}] \le 22$.

Thus, our upper bound for degrees of fields from $\mathcal{F}\Gamma_3^{(4)}(14)$ is 53.

5.4 Fields from $\mathcal{F}\Gamma_4^{(4)}(14)$

For $\Gamma_4^{(4)}(14)$ (see Figure 7), $k \ge 2$, $s, r \ge 3$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following possibilities: $s = 3$, $r = 3, 4, 5$; $s = 4, 5$, $r = 3$.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2\frac{\pi}{s},\ \cos^2\frac{\pi}{r},\ \cos^2\frac{\pi}{k},\ u^2,\ u\cos\frac{\pi}{s}\cos\frac{\pi}{k}\,.$$

This case can be considered as a specialization of the graph $\Gamma_1^{(4)}$ when we take $p = 2$. The determinant $d(u)$ of the Gram matrix is determined by the equality

$$\frac{-d(u)}{4} = u^2 + 4\cos\frac{\pi}{s}\cos\frac{\pi}{k}\,u + 4\cos^2\frac{\pi}{s} - 4\sin^2\frac{\pi}{k}\sin^2\frac{\pi}{r}.$$

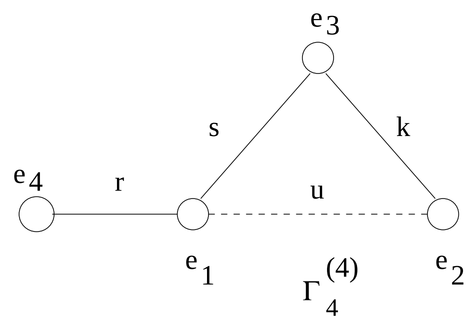

Figure 7: The graph $\Gamma_4^{(4)}$

Here

$$\widetilde{u} = u^2 + 4\cos\frac{\pi}{s}\cos\frac{\pi}{k}\,u + 4\cos^2\frac{\pi}{s}$$

is a totally positive algebraic integer (since minimum of this quadratic polynomial of u is equal to $4\cos^2\frac{\pi}{s}\sin^2\frac{\pi}{k}$) which belongs to $\mathbb{K}$, and $\Gamma_4^{(4)}$ is V-arithmetic if and only if

$$0 < \sigma(4\cos^2\frac{\pi}{s}\sin^2\frac{\pi}{k}) \le \sigma(\widetilde{u}) < \sigma(4\,\sin^2\frac{\pi}{r}\sin^2\frac{\pi}{k}) < 4$$

for any $\sigma : \mathbb{K} \to \mathbb{R}$ which is different from identity $\sigma^{(+)}$. For $\sigma^{(+)}$, we have

$$4 < \sigma^{(+)}(\widetilde{u}) < 14^2 + 4\cdot 14 + 4 = 16^2.$$

It follows that $\mathbb{K} = \mathbb{Q}(\widetilde{u})$. We can apply Theorems 13 and 14 to $\mathbb{F} = \mathbb{Q}(\cos^2\frac{\pi}{s}, \cos^2\frac{\pi}{r},\ \cos^2\frac{\pi}{k})$ and $\alpha = \widetilde{u}$ to estimate $[\mathbb{K} : \mathbb{Q}]$. We can take $M = [\mathbb{F} : \mathbb{Q}]$, $B = \sqrt{|\mathrm{discr}\ \mathbb{F}|}$,

$$R = \sqrt{N_{\mathbb{F}/\mathbb{Q}}\left((\sin^2\frac{\pi}{r} - \cos^2\frac{\pi}{s})\sin^2\frac{\pi}{k}\right)},\quad S = \frac{2\cdot 16^2\cdot e}{4(\sin^2\frac{\pi}{r} - \cos^2\frac{\pi}{s})\sin^2\frac{\pi}{k}}\,.$$

Then $[\mathbb{K} : \mathbb{Q}] \le MN$ where N is the least natural solution of the inequality

$$N\ln(1/R) - M\ln(2N+2) - \ln B \ge \ln S.$$

It follows that $[\mathbb{K} : \mathbb{Q}] \le 31$ for $2 \le k \le 6$ with the maximum bound 31 for $k = 2$ and $s = r = 3$.

When $k \ge 7$, additionally, we should use the inequality

$$\frac{16^2 N_{\mathbb{K}/\mathbb{Q}}(4\,\sin^2\frac{\pi}{r}\sin^2\frac{\pi}{k})}{4\,\sin^2\frac{\pi}{r}\sin^2\frac{\pi}{k}} > N_{\mathbb{K}/\mathbb{Q}}(\widetilde{u}) \ge 1 \tag{21}$$

which follows from considerations above. This additional arguments will be very similar to much more difficult case of $\Gamma_5^{(4)}(14)$ which we will consider below.

Our upper bound for degrees of fields from $\mathcal{F}\Gamma_4^{(4)}(14)$ will be 120 (look at the end of the next section 5.5).

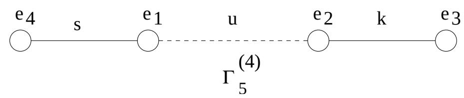

Figure 8: The graph $\Gamma_5^{(4)}$

5.5 Fields from $\mathcal{F}\Gamma_5^{(4)}(14)$

For $\Gamma_5^{(4)}(14)$ (see Figure 8), $k \geq s \geq 3$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2 \frac{\pi}{k},\ \cos^2 \frac{\pi}{s},\ u^2\ .$$

The determinant $d(u)$ of the Gram matrix is determined by the equality

$$-\frac{d(u)}{4} = u^2 - 4\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}.$$

The $\Gamma_5^{(4)}$ is V-arithmetic if and only if

$$\sigma(u^2) < \sigma(4\ \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}) < 4$$

for any $\sigma : \mathbb{K} \to \mathbb{R}$ which is different from identity $\sigma^{(+)}$. For $\sigma^{(+)}$, we have

$$4 < \sigma^{(+)}(u^2) < 14^2.$$

We can apply Theorems 13 and 14 to $\mathbb{F} = \mathbb{Q}(\cos^2 \frac{\pi}{k},\ \cos^2 \frac{\pi}{s})$ and $\alpha = u^2$ to estimate $[\mathbb{K} : \mathbb{Q}]$. We can take $M = [\mathbb{F} : \mathbb{Q}]$, $B = \sqrt{|\text{discr } \mathbb{F}|}$,

$$R = \sqrt{N_{\mathbb{F}/\mathbb{Q}} \left(\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s} \right)}\ , \quad S = \frac{14^2 \cdot e}{2(\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s})}.$$

Then $[\mathbb{K} : \mathbb{Q}] \leq MN_1$ where N_1 is the least natural solution of the inequality

$$N_1 \ln(1/R) - M \ln(2N_1 + 2) - \ln B \geq \ln S.$$

For each fixed pair $k \geq s \geq 3$ we can do it and obtain an estimate of the degree $[\mathbb{K} : \mathbb{Q}]$. Let us call this method as the *Method A*.

From our considerations above, we obtain the inequality

$$\frac{14^2 N_{\mathbb{K}/\mathbb{Q}}(4\ \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s})}{4\ \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}} > N_{\mathbb{K}/\mathbb{Q}}(u^2) \geq 1\,. \tag{22}$$

We use the following elementary facts about cyclotomic fields. Let $\mathbb{F}_l = \mathbb{Q}(\cos^2(\pi/l))$ where $l \geq 3$. Then

$$[\mathbb{F}_l : \mathbb{Q}] = \varphi(l)/2 \tag{23}$$

where φ is the Euler function. Let $\mathbb{F}_{l,m} = \mathbb{Q}(\cos^2(\pi/l), \cos^2(\pi/m))$ where $l, m \geq 3$. Then

$$[\mathbb{F}_{l,m} : \mathbb{Q}] = \frac{\varphi([l,m])}{2\rho(l,m)} \tag{24}$$

where $[\ ,\ ]$ denotes the least common multiple and $\rho(l,m) = 2$ if $(l,m)|2$, and $\rho(l,m) = 1$ otherwise. Here $(\ ,\ )$ denotes the greatest common divisor.

We have for $l \geq 3$

$$N_{\mathbb{F}_l/\mathbb{Q}}(4\sin^2(\pi/l)) = \gamma(l) = \begin{cases} p & \text{if } l = p^t > 2 \text{ where } p \text{ is prime,} \\ 1 & \text{otherwise.} \end{cases} \tag{25}$$

We denote $\mathbb{F} = \mathbb{F}_{k,s}$, $N = [\mathbb{K} : \mathbb{Q}]$, $n = [k,s]$, $[\mathbb{K} : \mathbb{F}_{k,s}] = m$. By (25), we have

$$N_{\mathbb{K}/\mathbb{Q}}(4\ \sin^2\frac{\pi}{k}\sin^2\frac{\pi}{s}) = \frac{N_{\mathbb{K}/\mathbb{Q}}(4\ \sin^2\frac{\pi}{k})N_{\mathbb{K}/\mathbb{Q}}(4\ \sin^2\frac{\pi}{s})}{N_{\mathbb{K}/\mathbb{Q}}(4)} =$$

$$\frac{\gamma(k)^{2N/\varphi(k)}\gamma(s)^{2N/\varphi(s)}}{4^N} = \left(\frac{\gamma(k)^{2/\varphi(k)}\gamma(s)^{2/\varphi(s)}}{4}\right)^N$$

where $(\varphi(n)/2\rho(k,s))|N$.

Hence, by (22), (23), (24), (25), we obtain

$$N\left(\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(s)}{\varphi(s)}\right) < \ln 7 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{s}, \quad \varphi(n)/2\rho(k,s)|N. \tag{26}$$

By exact formulae for $\gamma(k)$ and $\varphi(k)$, it is easy to prove that there exists only finite number of *exceptional pairs* (k,s) such that $k \geq s \geq 3$ and

$$\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(s)}{\varphi(s)} \leq 0. \tag{27}$$

They are $s = 3$, $k = 3, 4, 5, 7, 8, 9, 11, 13, 17, 19$; $s = 4$, $k = 4, 5$; $s = 5$, $k = 5, 7$.

For non-exceptional $k \geq s$ we get

$$\frac{\ln 7 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{s}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(s)}{\varphi(s)}} > \frac{\varphi([k,s])}{2\rho(k,s)}, \tag{28}$$

$$[\mathbb{K} : \mathbb{F}_{k,s}] \leq \left[\frac{\ln 7 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{s}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(s)}{\varphi(s)}} \Big/ \frac{\varphi([k,s])}{2\rho(k,s)}\right], \tag{29}$$

and

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left[\frac{\ln 7 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{s}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(s)}{\varphi(s)}} \Big/ \frac{\varphi([k,s])}{2\rho(k,s)}\right] \cdot \frac{\varphi([k,s])}{2\rho(k,s)}\ . \tag{30}$$

Let us show that there exists only a finite number of non-exceptional pairs $k \geq s \geq 3$ satisfying the inequality (28).

By exact formulae for $\gamma(l)$ and $\varphi(l)$, it is easy to find the minimum of $\ln 2 - \ln\gamma(k)/\varphi(k) - \ln\gamma(s)/\varphi(s)$ for all non-exceptional pairs. It is achieved for $k = 23$, $s = 3$, and it is equal to $ln2 - log(23)/22 - log(3)/2 \geq 0.00131857$. Since $x \geq \sin x$ for small x, $\rho(k,s) \leq 2$ and $\varphi([k,s]) \geq \varphi(k)$, we obtain $ln7 - 2ln\pi + 2ln(k) \geq 0.00131857 \cdot \varphi(k)/4$. Using the trivial estimate $\varphi(k) \geq \sqrt{k-2}$, we obtain $ln7 - 2ln\pi + 2ln(k) \geq 0.0003296425\sqrt{k-2}$. It follows that $s \leq k < 2.1 \cdot 10^{10}$. It follows the finiteness.

Using a better estimate $\varphi(k) \geq Ck/ln(ln(k))$, one can get a better estimate for k. One can take $C = \varphi(6)log(log(6))/6 \geq 0.19439$ for $k \geq 6$. See [32]. It follows, $s \leq k < 10^7$.

It follows, that all non-exceptional pairs $k \geq s$ satisfying (28) can be found (using a computer). Using (29) and (30), the bounds for $[\mathbb{K} : \mathbb{F}_{k,s}]$ and for $N = [\mathbb{K} : \mathbb{Q}] = [\mathbb{K} : \mathbb{F}_{k,s}][\mathbb{F}_{k,s} : \mathbb{Q}]$ can be found for each such a pair. This we call the *Method B*.

If (30) gives a poor bound for N because of the bound (29) for $[\mathbb{K} : \mathbb{F}_{k,s}]$ is poor, we can improve the bound for $[\mathbb{K} : \mathbb{F}_{k,s}]$ using the Method A above. Also we can apply the Method A to all exceptional pairs $k \geq s$.

As a result, we obtain that $[\mathbb{K} : \mathbb{Q}] \leq 120$. This is achieved for $k = 31$, $s = 3$. In this case, (30) gives $[\mathbb{K} : \mathbb{Q}] \leq 165$, and (29) gives $[\mathbb{K} : \mathbb{F}_{31,3}] \leq 11$. But, the Method A improves the last estimate and gives $[\mathbb{K} : \mathbb{F}_{31,3}] \leq 8$. Thus, we obtain $[\mathbb{K} : \mathbb{Q}] \leq 15 \cdot 8 = 120$ since $[\mathbb{F}_{31,3} : \mathbb{Q}] = 15$.

For all other cases when (30) gives a bound $[\mathbb{K} : \mathbb{Q}] \leq t$ where $t > 120$, we can similarly improve this bound using the Method A applied to $\mathbb{F} = \mathbb{F}_{k,s}$. For example, for $k = 23$ and $s = 3$, the inequality (30) gives $N = [\mathbb{K} : \mathbb{Q}] \leq 3091$, and (29) gives $[\mathbb{K} : \mathbb{F}_{23,3}] \leq 281$. Applying the Method A, we obtain $[\mathbb{K} : \mathbb{F}_{23,3}] \leq 8$ and $[\mathbb{K} : \mathbb{Q}] \leq 11 \cdot 8 = 88$. Surprisingly, this strategy works in all bad cases.

Thus, our upper bound for degrees of fields from $\mathcal{F}\Gamma_5^{(4)}(14)$ is 120.

Now, considering the graphs $\Gamma_4^{(4)}(14)$ again, from (21), we similarly get the inequalities

$$\frac{\ln 8 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{r}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(r)}{\varphi(r)}} > \frac{\varphi([k,r])}{2\rho(k,r)}, \tag{31}$$

$$[\mathbb{K} : \mathbb{F}_{k,r}] \leq \left[\frac{\ln 8 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{r}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(r)}{\varphi(r)}} \Big/ \frac{\varphi([k,r])}{2\rho(k,r)}\right], \tag{32}$$

and

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left[\frac{\ln 8 - \ln\sin\frac{\pi}{k} - \ln\sin\frac{\pi}{r}}{\ln 2 - \frac{\ln\gamma(k)}{\varphi(k)} - \frac{\ln\gamma(r)}{\varphi(r)}} \Big/ \frac{\varphi([k,r])}{2\rho(k,r)}\right] \frac{\varphi([k,r])}{2\rho(k,r)}\ . \tag{33}$$

for non-exceptional $k \geq r$ where $r = 3, 4, 5$ and $k \geq 6$ (thus, one should replace s by r and 7 by 8 in our considerations of $\Gamma_5^{(4)}(14)$ above).

Then exactly the same considerations as for $\Gamma_5^{(4)}(14)$ show that $[\mathbb{K} : \mathbb{Q}] \leq 120$ for all fields $\mathbb{K}$ from $\Gamma_4^{(4)}(14)$ where 120 is achieved for $k = 31$ and $r = 3$.

This finishes the proof of Theorem 4.

6 A mirror symmetric finiteness conjecture about reflective automorphic forms on Hermitian symmetric domains of type IV

In [10]—[13] some finiteness results and conjectures about so called *reflective automorphic forms on symmetric domains of type IV* (in classification by Cartan) were obtained and formulated. They were considered as mirror symmetric statements to finiteness results about arithmetic hyperbolic reflection groups over $\mathbb{Q}$ and corresponding reflective hyperbolic lattices over $\mathbb{Z}$.

Now finiteness results about arithmetic hyperbolic reflection groups are established in full generality. Moreover, results of this paper can be considered as some steps to classification in the future. Respectively, it would be interesting to extend and formulate the corresponding finiteness conjecture about reflective automorphic forms on symmetric domains of type IV in full generality too. Let us do it.

Let $\mathbb{K}$ be a totally real algebraic number field and $\mathbb{O}$ its ring of algebraic integers.

We recall that a lattice L over $\mathbb{K}$ is a finitely generated torsion-free $\mathbb{O}$-module L with a symmetric bilinear form defined on L with values in $\mathbb{O}$. Here $\mathbb{K}$ is called the ground field of L, the number $\dim L \otimes_{\mathbb{O}} \mathbb{K}$ is called the rank of S, and the absence of torsion means that $S \subset L \otimes_{\mathbb{O}} \mathbb{K}$. We let $x \cdot y$ denote the value of the bilinear form on L on the pair of elements $x, y \in L$, and we let x^2 denote $x \cdot x$.

A lattice S is said to be *hyperbolic* if the bilinear form $S \otimes_{\mathbb{O}} \mathbb{R}$ over $\mathbb{R}$ is indefinite for exactly one embedding $\sigma^{(+)} : \mathbb{K} \to \mathbb{R}$, and it is hyperbolic under this embedding, i. e., it has signature $(1, t_{(-)})$. For $\operatorname{rank} S \geq 3$ we let $\mathcal{L}(S)$ denote the hyperbolic space of the dimension $\operatorname{rank} S - 1$ which is canonically associated with the hyperbolic lattice S (and with the form $S \otimes_{\mathbb{O}} \mathbb{R}$ under the embedding $\sigma^{(+)}$):

$$\mathcal{L}(S) = \{\mathbb{R}^+ x \subset S \otimes_{\mathbb{O}} \mathbb{R} \mid x^2 > 0\}_0$$

where 0 means that we take a connected component. The automorphism group $O^+(S)$ of the hyperbolic lattice S is discrete and arithmetic in $\mathcal{L}(S)$, and it has fundamental domain of finite volume.

A lattice T is said to be *of IV type* if the bilinear form $S \otimes_{\mathbb{O}} \mathbb{R}$ over $\mathbb{R}$ is indefinite for exactly one embedding $\sigma^{(+)} : \mathbb{K} \to \mathbb{R}$, and it has signature $(2, t_{(-)})$ for this embedding. For $\operatorname{rank} T \geq 3$ we let $\Omega(T)$ denote the Hermitian symmetric domain of type IV (of dimension $\operatorname{rank} T - 2$) which is canonically associated with the IV type lattice T (and with the form $T \otimes_{\mathbb{O}} \mathbb{R}$ under the embedding $\sigma^{(+)}$):

$$\Omega(T) = \{\mathbb{C}\omega \subset T \otimes \mathbb{C} \mid \omega^2 = 0, \omega \cdot \overline{\omega} > 0\}_0 \ .$$

The automorphism group $O^+(T)$ of the IV type lattice T is discrete and arithmetic in $\Omega(T)$, and it has fundamental domain of finite volume.

For both types of lattices (hyperbolic or of IV type) L we can define reflections as follows. Let $\delta \in L$, where $\sigma^{(+)}(\delta^2) < 0$ and $\delta^2 \mid 2(L \cdot \delta)$ (such elements

are called *roots* of L). Then the formula

$$s_\delta(x) = x - \frac{2(x \cdot \delta)}{\delta^2}\,\delta,\ x \in L,$$

defines an involution s_δ of the lattice L which is called reflection relative to the root δ of L. We let $W(L)$ denote the subgroup of $O(L)$ generated by all of the reflections of L. Geometrically, in hyperbolic (respectively IV type) case the reflections s_δ are precisely those automorphisms of L which act as reflections relative to hyperplanes of $\mathcal{L}(L)$ (respectively to quadratic divisors $D_\delta = \{\mathbb{C}\omega \in \Omega(T) \mid \omega \cdot \delta = 0\}$ of $\Omega(L)$). Obviously, they are orthogonal to the roots δ.

A hyperbolic lattice S of the rank no less than three is called *reflective* if $W(S)$ is a subgroup of $O(S)$ of finite index.

By Vinberg's arithmeticity criterion, any arithmetic hyperbolic reflection group W is a subgroup of finite index $W \subset W(S)$ for one of reflective hyperbolic lattices S. The ground field $\mathbb{K}$ of S then coincides with the ground field of W.

It was proved in [16] and [17] that for a fixed degree $N = [\mathbb{K} : \mathbb{Q}]$ of the ground field of S and fixed rank $S \geq 3$ there exists only finitely many reflective hyperbolic lattices S up to similarity (i.e. up to multiplication of the form of S by elements $k \in \mathbb{K}$). Since it is now established in full generality that the rank S and degree N are absolutely bounded, it follows that

there exist finitely many similarity classes of reflective hyperbolic lattices

(for all ranks ≥ 3 and for all ground fields together).

Now let T be a lattice of IV type. A holomorphic automorphic form Φ of a positive weight on $\Omega(T)$ which is automorphic relative to $O^+(T)$ is called *reflective automorphic form of the lattice* T if the divisor of Φ is union of quadratic divisors of $\Omega(T)$ which are orthogonal to roots δ of reflections s_δ of T. A lattice T of IV type is called *reflective* if it has at least one reflective automorphic form Φ.

Conjecture 15. *There exist finitely many similarity classes of IV type reflective lattices of rank at least 5 (for all ranks ≥ 5 and all ground fields together).*

For $\mathbb{K} = \mathbb{Q}$ and respectively $\mathbb{O} = \mathbb{Z}$ this conjecture was formulated in [10]–[13]. Even in this case, it seems, it is not established in full generality.

Some arithmetic hyperbolic reflection groups and some reflective automorphic forms and corresponding hyperbolic and IV type reflective lattices over $\mathbb{Z}$ are important in Borcherds proof [3] of Moonshine Conjecture by Conway and Norton [7] which had been first discovered by John McKay.

We hope that similar objects over arbitrary number fields will find similar astonishing applications in the future. At least, the results and conjectures of this paper show that they are very exceptional even in this very general setting.

7 Appendix: Hyperbolic numbers (the review of [17, Sec. 1])

7.1 Fekete's theorem

Here we review our results in [17, Sec.1] and correct some arithmetic mistakes (Theorems 1.1 and 1.2.2 in [17] which are similar to Theorems 17 and 19 here). This mistakes are unessential for results of [17].

The following important theorem, to which this section is devoted, was obtained by Fekete [9]. Although Fekete considered (as we know) the case of $\mathbb{Q}$ his method of proof can be immediately carried over to totally real algebraic number fields.

Theorem 16. *(M. Fekete). Suppose that $\mathbb{F}$ is a totally real algebraic number field, and to every embedding $\sigma : \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$ and the real number $\lambda_\sigma > 0$. Suppose that $\prod_\sigma \lambda_\sigma = 1$. Then for every non-negative integer n there exists a nonzero polynomial $P_n(T) \in \mathbb{O}[T]$ of degree no greater than n over the ring of integers $\mathbb{O}$ of $\mathbb{F}$ such that the following inequality holds for each σ:*

$$\max_{x\in[a_\sigma,b_\sigma]} |P_n^\sigma(x)| \le \lambda_\sigma |\operatorname{discr} \mathbb{F}|^{1/(2[\mathbb{F}:\mathbb{Q}])} 2^{n/(n+1)}(n+1) \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4} \right)^{n/(2[\mathbb{F}:\mathbb{Q}])}. \tag{34}$$

Proof. Suppose that $N = [\mathbb{F} : \mathbb{Q}]$ and that $\gamma_1, \dots \gamma_N$ is the basis for $\mathbb{O}$ over $\mathbb{Z}$. Suppose we are given a nonzero polynomial

$$P_n(T) = \sum_{i=0}^{n} \sum_{j=1}^{N} \alpha_{ij}\gamma_j T^i \in \mathbb{O}[T]$$

of degree no greater than n, where the $\alpha_{ij} \in \mathbb{Z}$ are not all zero. For every $\sigma : \mathbb{F} \to \mathbb{R}$ we consider the real functions $P_n^\sigma(x)$ on the interval $[a_\sigma, b_\sigma]$.

We make the change of variables

$$x = x(z) = \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z.$$

If z runs through $[0, \pi]$, then x runs through $[a_\sigma, b_\sigma]$. We also set $Q_n^\sigma(z) = P_n^\sigma(x(z))$.

Since $Q_n^\sigma(z)$ is an even trigonometric polynomial, it follows that

$$Q_n^\sigma(z) = \sum_{k=0}^{n} A_{k\sigma} \cos kz, \tag{35}$$

where

$$A_{k\sigma} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_n^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) \cos kz \, dz$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{N} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \cos kz \, dz \right) \alpha_{ij} \,,$$

if $k \geq 1$, and

$$A_{0\sigma} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) dz$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i dz \right) \alpha_{ij} \,.$$

Thus,

$$A_{k\sigma} = \sum_{i=0}^{n} \sum_{j=1}^{N} c_{k\sigma ij} \alpha_{ij} \tag{36}$$

are linear functions of the α_{ij}, where

$$c_{k\sigma ij} = \gamma_j^\sigma \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \cos kz \, dz \,,$$

if $k \geq 1$, and

$$c_{0\sigma ij} = \gamma_j^\sigma \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i dz \,.$$

We note that, because of these formulas, $c_{k\sigma ij} = 0$ for $i < k$, and

$$c_{k\sigma kj} = \gamma_j^\sigma \cdot 2 \left(\frac{b_\sigma - a_\sigma}{4} \right)^k \,, \quad \text{if } k \geq 1$$

($c_{0\sigma 0j} = \gamma_j^\sigma$). Hence, if we order the indices $k\sigma$ and ij lexicographically, we find that the matrix of the linear forms (36) is an upper block-triangular matrix with the shown above $N \times N$ matrices $(c_{0\sigma 0j})$ and $(c_{k\sigma kj})$, $1 \leq k \leq n$, on the diagonal. It follows that its determinant is equal to

$$\Delta = \det(\gamma_j^\sigma)^{n+1} \cdot 2^{Nn} \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4} \right)^{n(n+1)/2} \,.$$

Since $\prod_\sigma \lambda_\sigma^{n+1} = (\prod_\sigma \lambda_\sigma)^{n+1} = 1$, according to Minkowski's theorem on linear forms (see, for example, [5], [6]), there exist $\alpha_{ij} \in \mathbb{Z}$, not all zero, such that $|A_{k\sigma}| \leq \lambda_\sigma |\Delta|^{1/N(n+1)}$, and hence, by (35),

$$\max_z |Q_n^\sigma(z)| \leq \lambda_\sigma \cdot (n+1) |\Delta|^{1/N(n+1)}.$$

Taking into account that $\det(\gamma_j^\sigma)^2 = \operatorname{discr} \mathbb{F}$, we obtain the proof of the theorem. $\square$

Taking $\lambda_\sigma = 1$, we get a particular statement which we later use.

Theorem 17. *(M. Fekete). Suppose that $\mathbb{F}$ is a totally real algebraic number field, and to every embedding $\sigma : \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$. Then for every nonnegative integer n there exists a nonzero polynomial $P_n(T) \in \mathbb{O}[T]$ of degree no greater than n over the ring of integers $\mathbb{O}$ of $\mathbb{F}$ such that the following inequality holds for each σ:*

$$\max_{x\in[a_\sigma,b_\sigma]} |P_n^\sigma(x)| \leq |\operatorname{discr} \mathbb{F}|^{1/(2[\mathbb{F}:\mathbb{Q}])} 2^{n/(n+1)}(n+1) \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4}\right)^{n/(2[\mathbb{F}:\mathbb{Q}])}. \tag{37}$$

7.2 Hyperbolic numbers

The totally real algebraic integers $\{\alpha\}$ which we consider here are very similar to Pisot-Vijayaraghavan numbers [6], although the later are not totally real.

Theorem 18. *Let $\mathbb{F}$ be a totally real algebraic number field, and let each imbedding $\sigma : \mathbb{F} \to \mathbb{R}$ corresponds to an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$, where*

$$\prod_\sigma \frac{b_\sigma - a_\sigma}{4} < 1.$$

In addition, let the natural number m and the intervals $[s_1, t_1], \dots, [s_m, t_m]$ in $\mathbb{R}$ be fixed.

Then there exists a constant $N(s_i, t_i)$ such that, if α is a totally real algebraic integer and if the following inequalities hold for the imbeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i, \ \textit{for } \tau = \tau_1, \dots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq b_{\tau|\mathbb{F}} \ \textit{for } \tau \neq \tau_1, \dots \tau_m,$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).$$

Theorem 19. *Under the conditions of Theorem 18, $N(s_i, t_i)$ can be taken to be $N(s_i, t_i) = N(S)$, where $N(S)$ is the least natural number solution of the inequality*

$$nln(1/R) - M \ln(2n+2) - \ln B \geq \ln S. \tag{38}$$

Here

$$M = [\mathbb{F} : \mathbb{Q}], \quad R = \sqrt{\prod_\sigma \frac{b_\sigma - a_\sigma}{4}}, \tag{39}$$

$$B = \sqrt{|\operatorname{discr} \mathbb{F}|}, \quad S = \prod_{i=1}^m \left(2er_i(b_{\sigma_i} - a_{\sigma_i})^{-1}\right), \tag{40}$$

where $\sigma_i = \tau_i|\mathbb{F}$ *and* $r_i = \max\{|t_i - a_{\sigma_i}|,\ |b_{\sigma_i} - s_i|\}$.

Asymptotically,

$$N(s_i, t_i) \sim \frac{\ln S}{\ln(1/R)}.$$

Proof. We use the following statement.

Lemma 20. *Suppose that* $Q_n(T) \in \mathbb{R}[T]$ *is a non-zero polynomial over* $\mathbb{R}$ *of degree no greater than* $n > 0$, $a < b$ *and* $M_0 = \max_{[a,b]}\{|Q_n(x)|\}$. *Then for* $x \geq b$

$$|Q_n(x)| \leq \frac{M_0(x-a)^n n^n}{((b-a)/2)^n\, n!} <$$

$$\frac{M_0(x-a)^n e^n}{((b-a)/2)^n \sqrt{2\pi n}} < \frac{M_0(x-a)^n e^n}{((b-a)/2)^n}.$$

Proof. Let $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then we have the Lagrange interpolation formula

$$Q_n(x) = \sum_{i=0}^{n} Q_n(\alpha_i) F_i(x)$$

where

$$F_i(x) = \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)}{(\alpha_i-\alpha_0)(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)}.$$

Taking $\alpha_i = a + i(b-a)/n$, $0 \leq i \leq n$, we obtain for $x \geq b$ that

$$|Q_n(x)| \leq \frac{M_0(x-a)^n}{((b-a)/n)^n} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} = \frac{M_0(x-a)^n 2^n}{((b-a)/n)^n n!}.$$

By Stirling formula, $n! = \sqrt{2\pi n}(n/e)^n e^{\lambda_n}$ where $0 < \lambda_n < 1/(12n)$. Thus, $n^n/n! < e^n/\sqrt{2\pi n} < e^n$. It follows the statement. $\square$

We continue the proof of theorems.

For given n we consider the polynomial $P_n(T) \in \mathbb{O}[T]$ whose existence is ensured by Fekete's theorem 17. Setting $N = [\mathbb{F}(\alpha) : \mathbb{F}]$ and $M = [\mathbb{F} : \mathbb{Q}]$, we use Fekete's theorem and the lemma to conclude that

$$|\prod_{\tau} \tau(P_n(\alpha))| = \prod_{\tau} |P_n^{\tau}(\tau(\alpha))| = \prod_{\tau \neq \tau_i} P_n^{\tau}(\tau(\alpha)) \prod_{i=1}^{m} |P_n^{\tau_i}(\tau_i(\alpha))|$$

$$\leq \prod_{\tau \neq \tau_i} \max_{[a_{\tau|\mathbb{F}}, b_{\tau|\mathbb{F}}]} |P_n^{\tau}(x)| \prod_{i=1}^{m} \max_{[s_i, t_i]} |P_n^{\sigma_i}(x)|$$

$$\leq \left(|\operatorname{discr} \mathbb{F}|^{1/(2M)} \cdot 2 \cdot (n+1) R^{n/M}\right)^{NM} \prod_{i=1}^{m} \frac{r_i^n e^n}{((b_{\sigma_i} - a_{\sigma_i})/2)^n}$$

$$= R^{nN} B^N \cdot S^n \cdot (2n+2)^{MN}.$$

Since $R < 1$, there exists n_0 large enough so that

$$R^{n_0} \cdot B \cdot (2n_0 + 2)^M \le \frac{1}{S}. \tag{41}$$

Then if $N > n_0$, we find that

$$R^{n_0 N} \cdot B^N \cdot S^{n_0} (2n_0 + 2)^{MN} \le S^{n_0 - N} < 1,$$

since $S > 1$. From this and the above chain of inequalities we have

$$|\prod_\tau \tau(P_{n_0}(\alpha))| < 1.$$

But

$$\prod_\tau \tau(P_{n_0}(\alpha)) = N_{\mathbb{F}(\alpha)/\mathbb{Q}}(P_{n_0}(\alpha)) \in \mathbb{Z},$$

and hence $P_{n_0}(\alpha) = 0$. Consequently, $N \le n_0$, and we have obtained a contradiction. We have thereby proved that $N \le n_0$, where n_0 is a natural number solution of (41). The inequality (41) is obviously equivalent to

$$n_0 \ln(1/R) - M \ln(2n_0 + 2) - \ln B \ge \ln S,$$

and this completes the proof of the theorems. □

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